

SELMER COMPLEXES AND THE p -ADIC HODGE THEORY

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INTRODUCTION

0.1. The Selmer group. In this paper we discuss some applications of p -adic Hodge theory to the algebraic formalism of Iwasawa theory. Fix an odd prime number p . For every finite set S of primes containing p we denote by $\mathbf{Q}^{(S)}/\mathbf{Q}$ the maximal algebraic extension of \mathbf{Q} unramified outside $S \cup \{\infty\}$ and set $G_{\mathbf{Q},S} = \text{Gal}(\mathbf{Q}^{(S)}/\mathbf{Q})$. Let $G_{\mathbf{Q}_v} = \text{Gal}(\overline{\mathbf{Q}_v}/\mathbf{Q}_v)$ denote the decomposition group at v and I_v its inertia subgroup.

Let M be a pure motive over \mathbf{Q} . The complex L -function $L(M, s)$ is a Dirichlet series which converges for $s \gg 0$ and is expected to admit a meromorphic continuation on the whole \mathbf{C} with a functional equation of the form

$$\Gamma(M, s)L(M, s) = \varepsilon(M, s)\Gamma(M^*(1), -s)L(M^*(1), -s).$$

where $\Gamma(M, s)$ is a product of Γ -factors explicitly defined in terms of the Hodge structure of M and $\varepsilon(M, s)$ is a factor of the form $\varepsilon(M, s) = ab^s$ defined in terms of local constants (see for example [26]).

Let V denote the p -adic realisation of M . We consider V as a p -adic representation of the Galois group $G_{\mathbf{Q},S}$ where S contains p , ∞ and the places where M has bad reduction. We will write $H_S^*(\mathbf{Q}, V)$ and $H^*(\mathbf{Q}_v, V)$ for the continuous cohomology of $G_{\mathbf{Q},S}$ and $G_{\mathbf{Q}_v}$ respectively. The Bloch–Kato Selmer group $H_f^1(\mathbf{Q}, V)$ is defined as

$$(1) \quad H_f^1(\mathbf{Q}, V) = \ker \left(H_S^1(\mathbf{Q}, V) \rightarrow \bigoplus_{v \in S} \frac{H^1(\mathbf{Q}_v, V)}{H_f^1(\mathbf{Q}_v, V)} \right)$$

where the "local conditions" $H_f^1(\mathbf{Q}_v, V)$ are given by

$$(2) \quad H_f^1(\mathbf{Q}_v, V) = \begin{cases} \ker(H^1(\mathbf{Q}_v, V) \rightarrow H^1(I_v, V)) & \text{if } v \neq p, \\ \ker(H^1(\mathbf{Q}_p, V) \rightarrow H^1(\mathbf{Q}_p, V \otimes \mathbf{B}_{\text{cris}})) & \text{if } v = p \end{cases}$$

(see [13]). Here \mathbf{B}_{cris} is the ring of crystalline periods of Fontaine [24]. Beilinson's conjecture (in the formulation of Bloch and Kato [13]) relates the rank of $H_f^1(\mathbf{Q}, V^*(1))$ to the order of vanishing of $L(M, s)$, namely one expects that

$$\text{ord}_{s=0} L(M, s) = \dim_{\mathbf{Q}_p} H_f^1(\mathbf{Q}, V^*(1)) - \dim_{\mathbf{Q}_p} H^0(\mathbf{Q}, V^*(1)).$$

0.2. p -adic L -functions. Assume that V satisfies the following conditions (see Section 2.1 below).

C1) $H_S^0(\mathbf{Q}, V) = H_S^0(\mathbf{Q}, V^*(1)) = 0$.

C2) The restriction of V on the decomposition group at p is semistable in the sense of Fontaine [25]. We denote by $\mathbf{D}_{\text{st}}(V)$ the semistable module associated to V . Recall that $\mathbf{D}_{\text{st}}(V)$ is a filtered \mathbf{Q}_p -vector space equipped

with a linear Frobenius $\varphi : \mathbf{D}_{\text{st}}(V) \rightarrow \mathbf{D}_{\text{st}}(V)$ and a monodromy operator $N : \mathbf{D}_{\text{st}}(V) \rightarrow \mathbf{D}_{\text{st}}(V)$ such that $N\varphi = p\varphi N$. Let $\mathbf{D}_{\text{cris}}(V) = \mathbf{D}_{\text{st}}(V)^{N=0}$.

C3) $\mathbf{D}_{\text{cris}}(V)^{\varphi=1} = 0$.

We define the notion of a regular submodule of $\mathbf{D}_{\text{st}}(V)$ which plays the key role in this paper. It was first introduced by Perrin-Riou [47] in the crystalline case. See also [5] and [7]. Eventually replacing M by $M^*(1)$ we will assume that M is of weight $\text{wt}(M) \leq -1$. We consider two cases:

The weight ≤ -2 case. Let $\text{wt}(M) \leq -2$. We expect that in this case the localization map

$$H_f^1(\mathbf{Q}, V) \rightarrow H_f^1(\mathbf{Q}_p, V)$$

is injective. The condition **C3)** implies that the Bloch–Kato exponential map

$$\exp_V : \mathbf{D}_{\text{st}}(V) / \text{Fil}^0 \mathbf{D}_{\text{st}}(V) \rightarrow H_f^1(\mathbf{Q}_p, V)$$

is an isomorphism and we denote by $\log_V : H_f^1(\mathbf{Q}_p, V) \rightarrow \mathbf{D}_{\text{st}}(V) / \text{Fil}^0 \mathbf{D}_{\text{st}}(V)$ its inverse. A (φ, N) -submodule D of $\mathbf{D}_{\text{st}}(V)$ is regular if $D \cap \text{Fil}^0 \mathbf{D}_{\text{st}}(V) = \{0\}$ and the composition of the localisation map with \log_V induces an isomorphism

$$r_{V,D} : H_f^1(\mathbf{Q}, V) \rightarrow \mathbf{D}_{\text{st}}(V) / (\text{Fil}^0 \mathbf{D}_{\text{st}}(V) + D).$$

Note that the map $r_{V,D}$ should be closely related to the syntomic regulator.

The weight -1 case. Let $\text{wt}(M) = -1$. In this case we say that a (φ, N) -submodule D of $\mathbf{D}_{\text{st}}(V)$ is regular if

$$\mathbf{D}_{\text{st}}(V) = D \oplus \text{Fil}^0 \mathbf{D}_{\text{st}}(V)$$

as vector spaces. Each regular D gives a splittings of the Hodge filtration on $\mathbf{D}_{\text{st}}(V)$ and therefore defines a p -adic height pairing

$$\langle \cdot, \cdot \rangle_{V,D} : H_f^1(\mathbf{Q}, V) \times H_f^1(\mathbf{Q}, V^*(1)) \rightarrow \mathbf{Q}_p$$

which is expected to be non-degenerate [42], [43], [45].

The theory of Perrin-Riou [47] suggests that to each regular D one can associate a p -adic L -function $L_p(M, D, s)$ interpolating rational parts of special values $L(M, s)$. The interpolation formula relating special values of the complex and the p -adic L -functions at $s = 0$ should have the form

$$\frac{L_p^*(M, D, 0)}{\Omega_p(M, D)} = \mathcal{E}(V, D) \frac{L^*(M, 0)}{\Omega_\infty(M)}.$$

Here $\Omega_\infty(M)$ is the irrational part of $L^*(M, 0)$ predicted by Beilinson–Deligne’s conjectures and $\Omega_p(M, D)$ is essentially the determinant of $r_{V,D}$

(in the weight ≤ -2 case) or the determinant of $\langle, \rangle_{V,D}$ (in the weight -1 case). Finally $\mathcal{E}(V, D)$ is an Euler-like factor. Its explicit form in the crystalline case was conjectured by Perrin-Riou [47], Chapitre 4 (see also [16] Conjecture 2.7). Namely, let $E_p(M, X) = \det(1 - \phi X | \mathbf{D}_{\text{cris}}(V))$. Note that $E_p(M, p^{-s})$ is the Euler factor of $L(M, s)$ at p . One expects that

$$\mathcal{E}(V, D) = E_p(M, 1) \det \left(\frac{1 - p^{-1} \phi^{-1}}{1 - \phi} | D \right) \quad \text{if } V \text{ is crystalline at } p.$$

0.3. Selmer complexes. We introduce basic notation of the Iwasawa theory. Let μ_{p^n} denote the group of p^n -th roots of unity and $\mathbf{Q}^{\text{cyc}} = \bigcup_{n=1}^{\infty} \mathbf{Q}(\mu_{p^n})$. We fix a system $(\zeta_{p^n})_{n \geq 0}$ of primitive p^n -th roots of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all n . The Galois group $\Gamma = \text{Gal}(\mathbf{Q}^{\text{cyc}}/\mathbf{Q})$ is canonically isomorphic to \mathbf{Z}_p^* via the cyclotomic character

$$\chi : \Gamma \rightarrow \mathbf{Z}_p^*, \quad g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)} \quad \text{for all } g \in \Gamma.$$

Let $\Delta = \text{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ and $\Gamma_0 = \text{Gal}(\mathbf{Q}^{\text{cyc}}/\mathbf{Q}(\mu_p))$. We denote by $F_{\infty} = \mathbf{Q}(\mu_{p^{\infty}})^{\Delta}$ the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q} and set $F_n = \mathbf{Q}(\mu_{p^{n+1}})^{\Delta}$, $n \in \mathbf{N}$. Fix a generator $\gamma \in \Gamma$ and set $\gamma_0 = \gamma^{p-1} \in \Gamma_0$. Let Λ denote the Iwasawa algebra $\mathbf{Z}_p[[\Gamma_0]]$. The choice of γ fixes an isomorphism $\Lambda \simeq \mathbf{Z}_p[[X]]$ such that $\gamma_0 \mapsto 1 + X$. Let \mathcal{H} denote the ring of power series that converge on the open unit disc. We consider Λ as a subring of \mathcal{H} via the isomorphism $\Lambda \simeq \mathbf{Z}_p[[X]]$. Note that \mathcal{H} is the large Iwasawa algebra introduced in [46].

Let T be \mathbf{Z}_p -lattice of V stable under the action of $G_{\mathbf{Q},S}$. Let $T \otimes \Lambda^l$ denote the tensor product $T \otimes \Lambda$ equipped with the diagonal action of $G_{\mathbf{Q},S}$ and the action of Λ given by the canonical involution $\iota : \Lambda \rightarrow \Lambda$

$$g(a \otimes \lambda) = a \otimes \iota(g)(\lambda) = a \otimes g^{-1} \lambda, \quad g \in \Gamma.$$

Let $C^{\bullet}(G_{\mathbf{Q},S}, T \otimes \Lambda^l)$ (respectively $C^{\bullet}(G_{\mathbf{Q}_v}, T \otimes \Lambda^l)$) be the complex of continuous cochains of $G_{\mathbf{Q},S}$ (respectively $G_{\mathbf{Q}_v}$) with coefficients in $T \otimes \Lambda^l$. We denote by $\mathbf{R}\Gamma_{\text{Iw},S}(\mathbf{Q}, T)$ and $\mathbf{R}\Gamma_{\text{Iw},S}(\mathbf{Q}_v, T)$ these complexes viewed as objects of the bounded derived category $D^b(\Lambda)$ of Λ -modules. Set $H_{\text{Iw},S}^i(\mathbf{Q}, T) = \mathbf{R}^i\Gamma_{\text{Iw},S}(\mathbf{Q}, T)$ and $H_{\text{Iw}}^i(\mathbf{Q}_v, T) = \mathbf{R}^i\Gamma_{\text{Iw}}(\mathbf{Q}_v, T)$. From Shapiro's lemma it follows that

$$H_{\text{Iw},S}^i(\mathbf{Q}, T) = \varprojlim_n H_S^i(F_n, T), \quad H_{\text{Iw}}^i(\mathbf{Q}_v, T) = \varprojlim_n H_S^i(F_{n,v}, T)$$

as Λ -modules. In [43] Nekovář considers diagrams of the form

$$(3) \quad \begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{Iw},S}(\mathbf{Q}, T) & \longrightarrow & \bigoplus_{v \in S} \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, T) \\ & & \uparrow \\ & & \bigoplus_{v \in S} U_v^\bullet(T). \end{array}$$

The cone of this diagram can be viewed as the derived version of Selmer groups where the complexes $U_v^\bullet(T)$ play the role of local conditions. For $v \neq p$ the natural choice of $U_v^\bullet(T)$ is to put $U_v^\bullet(T) = \mathbf{R}\Gamma_{\mathrm{Iw},f}(\mathbf{Q}_v, T)$ where

$$\mathbf{R}\Gamma_{\mathrm{Iw},f}(\mathbf{Q}_v, T) = \left[T^{I_v} \otimes \Lambda^t \xrightarrow{\mathrm{Fr}_v - 1} T^{I_v} \otimes \Lambda^t \right]$$

where Fr_v denotes the geometric Frobenius and the terms are placed in degrees 0 and 1. Note that the cohomology groups of

$$\mathbf{R}\Gamma_{\mathrm{Iw},f}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{Q}_p = \left[V^{I_v} \xrightarrow{\mathrm{Fr}_v - 1} V^{I_v} \right]$$

are $H^0(\mathbf{Q}_v, V)$ and $H_f^1(\mathbf{Q}_v, V)$. The local conditions at p are more delicate to define. First assume that V satisfies the Panchishkin condition i.e. the restriction of V on the decomposition group at p has a subrepresentation $F^+V \subset V$ such that

$$\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(F^+V) \oplus \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$$

as vector spaces. Set $U_p^\bullet(T) = \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_p, F^+T)$. Then $U_p^\bullet(T)$ is the derived version of Greenberg's local conditions [27] and the cohomology of Nekovář's Selmer complex is closely related to the Pontriagin dual of Greenberg's Selmer group [43]. If we assume in addition that $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} = 0$ then $H_f^1(\mathbf{Q}_p, V) = \ker(H^1(\mathbf{Q}_p, V) \rightarrow H^1(I_p, V/F^+V))$ and one can compare Greenberg's and Bloch–Kato's Selmer groups. Roughly speaking, in this case different definitions lead to pseudo-isomorphic Λ -modules which have therefore the same characteristic series in the case they are Λ -torsion (see [19], [43], Chapter 9 and [44]). If $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} \neq 0$ the situation is more complicated. The analytic counterpart of this problem is the phenomenon of extra zeros of p -adic L -functions studied in [28], [5], [7].

We no longer assume that V satisfies the Panchishkin condition. The theory of (φ, Γ) -modules (see [22], [15], [9]) associates to V a finitely generated free module $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ over the Robba ring \mathcal{R} equipped with a semilinear actions of the group Γ and a Frobenius φ which commute with each other. The category of (φ, Γ) -modules has a nice cohomology theory whose formal properties are very close to properties of local Galois cohomology [31],

[32], [41]. In particular, $H^*(\mathbf{D}_{\text{rig}}^\dagger(V))$ is canonically isomorphic¹ to the continuous Galois cohomology $H^*(\mathbf{Q}_p, V)$. Moreover, to each (φ, Γ) -module \mathbf{D} one can associate the complex

$$\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}) = \left[\mathbf{D} \xrightarrow{\psi-1} \mathbf{D} \right]^{\Delta=0}$$

where ψ is the left inverse to φ (see [14]) and the first term is placed in degree 1. We will write $H_{\text{Iw}}^*(\mathbf{D})$ for the cohomology of $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D})$. The action of Γ_1 induces a natural structure of \mathcal{H} -module on $\mathbf{D}^{\Delta=0}$. From a general result of Pottharst ([50], Theorem 2.8) it follows that

$$H_{\text{Iw}}^i(\mathbf{Q}_p, T) \otimes_{\Lambda} \mathcal{H} \xrightarrow{\sim} H_{\text{Iw}}^i(\mathbf{D}_{\text{rig}}^\dagger(V))$$

as \mathcal{H} -modules.

The approach to Iwasawa theory discussed in this paper is based on the observation that the (φ, Γ) -module of a semistable representation looks like an ordinary Galois representation, in particular it is a successive extension of (φ, Γ) -modules of rank 1. This was first pointed out by Colmez in [18] where the structure of trianguline (φ, Γ) -modules of rank 2 over \mathbf{Q}_p was studied in detail. Therefore in the non ordinary setting we can adopt Greenberg's approach working with (φ, Γ) -modules instead p -adic representations. This idea was used in [2] and [3] to study Selmer groups in families and in [5], [6], [7] to study extra-zeros of p -adic L -functions. Pottharst [48] started the general theory of Selmer complexes in this setting and related it to Perrin-Riou's theory [49].

0.4. The Main Conjecture. Fix a regular submodule D of $\mathbf{D}_{\text{st}}(V)$. By [11] one can associate to D a canonical (φ, Γ) -submodule \mathbf{D} of $\mathbf{D}_{\text{rig}}^\dagger(V)$. Consider the diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\text{Iw}, S}(\mathbf{Q}, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} & \longrightarrow & \bigoplus_{v \in S} \mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \\ & & \uparrow \\ & & \bigoplus_{v \in S} U_v^\bullet(V, D) \end{array}$$

in the derived category of \mathcal{H} -modules, where the local conditions $U_v^\bullet(V, D)$ are

$$U_v^\bullet(V, D) = \begin{cases} \mathbf{R}\Gamma_{\text{Iw}, f}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} & \text{if } v \neq p \\ \mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}) & \text{if } v = p. \end{cases}$$

¹Up to the choice of a generator of Γ .

Consider the Selmer complex associated to this data

$$(4) \quad \mathbf{R}\Gamma_{\mathrm{Iw}}(V, D) = \mathrm{cone} \left(\left(\mathbf{R}\Gamma_{\mathrm{Iw}, f}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \right) \oplus \left(\bigoplus_{v \in S} U_v^{\bullet}(V, D) \right) \rightarrow \bigoplus_{v \in S} \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \right) [-1].$$

To simplify notation define

$$H_{\mathrm{Iw}}^i(V, D) = \mathbf{R}^i\Gamma_{\mathrm{Iw}}(V, D).$$

We propose the following conjecture.

Main Conjecture. *Let M/\mathbf{Q} be a pure motive of weight ≤ -1 which does not contain submotives of the form $\mathbf{Q}(m)$. Assume that the p -adic realisation V of M satisfies the conditions **C1-3** above. Let D be a regular submodule of $\mathbf{D}_{\mathrm{st}}(V)$. Then*

- i) $H_{\mathrm{Iw}}^i(V, D) = 0$ for $i \neq 2$.*
- ii) $H_{\mathrm{Iw}}^2(V, D)$ is a coadmissible ² torsion \mathcal{H} -module and*

$$\mathrm{char}_{\mathcal{H}}(H_{\mathrm{Iw}}^2(V, D)) = (f_D)$$

where $L_p(M, D, s) = f_D(\chi(\gamma_0)^s - 1)$.

Remarks 1) Since $\mathcal{H}^* = \Lambda[1/p]^*$, our Main Conjecture determines f_D up to multiplication by a unit in $\Lambda[1/p]$.

2) Assume that M is critical and that V is ordinary at p . Then the ordinarity filtration $(F^i V)_{i \in \mathbf{Z}}$ provides the canonical regular module $D = \mathbf{D}_{\mathrm{st}}(F^1 V)$. It is expected that in this situation $f_D \in \Lambda$. In [27], Greenberg defined a cofinitely generated Λ -module (Greenberg's Selmer group) $S(F_{\infty}, V^*(1)/T^*(1))$ and conjectured that its Pontriagin dual $S(F_{\infty}, V^*(1)/T^*(1))^{\wedge}$ is related to the p -adic function $L_p(V, D, s)$ as follows

$$\mathrm{char}_{\Lambda} S(F_{\infty}, V^*(1)/T^*(1))^{\wedge} = f_D \Lambda$$

Using the results of [43] one can check that this agrees with our Main conjecture, but Greenberg's conjecture is more precise because it determines f_D up to multiplication by a unit in Λ . See [49] and Section 2.5 for more detail.

3) Assume that V is crystalline at p . In [47], for any regular D Perrin-Riou defined a free Λ -module $\mathbf{L}_{\mathrm{Iw}}(D, V)$ together with a canonical trivialisation

$$i_{V, D} : \mathbf{L}_{\mathrm{Iw}}(D, V) \xrightarrow{\sim} \mathcal{H}$$

²See Section 1.6 for the definition of a coadmissible module

See also [7] for the interpretation of Perrin-Riou's theory in terms of Selmer complexes. The Main Conjecture in the formulation of Perrin-Riou says that

$$i_{V,D}(\mathbf{L}_{\text{Iw}}(D,V)) = \Lambda f_D.$$

In [49], Pottharst proved that

$$i_{V,D}(\mathbf{L}_{\text{Iw}}(D,V)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \text{char } \mathcal{H} H_{\text{Iw}}^2(V,D)$$

i.e. for crystalline representations our conjecture is compatible with Perrin-Riou's theory.

4) Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a newform of weight k on $\Gamma_0(N)$ and let W_f be the p -adic representation associated to f . Then W_f can be seen as the p -adic realisation of a pure motive M_f of weight $k-1$. The representation W_f is crystalline if $(p,N)=1$ and semistable non crystalline if $p \parallel N$. Fix $1 \leq m \leq k-1$. Then the motive $M_f(m)$ is critical and we denote by $V = W_f(m)$ its p -adic realisation. The subspace $\text{Fil}^0 \mathbf{D}_{\text{st}}(V)$ is one dimensional. Eventually extending scalars to some E/\mathbf{Q}_p we can write $\mathbf{D}_{\text{st}}(V) = Ee_{\alpha} + Ee_{\beta}$ where $\varphi(e_{\alpha}) = \alpha e_{\alpha}$ and $\varphi(e_{\beta}) = \beta e_{\beta}$. If $(p,N)=1$ the subspaces $D_{\alpha} = Ee_{\alpha}$ and $D_{\beta} = Ee_{\beta}$ are regular and the corresponding p -adic L -functions $L(V, D_{\alpha}, s)$ and $L(V, D_{\beta}, s)$ are morally the usual p -adic L -functions associated to α and β . If $p \parallel N$, we have $Ne_{\beta} = e_{\alpha}$ and $Ne_{\alpha} = 0$ and D_{α} is the unique regular subspace of $\mathbf{D}_{\text{st}}(V)$. The results of Kato [35] have the following interpretation in terms of our theory. Assume that $(p,N)=1$ and set $L(V, D_{\alpha}, s) = f_{\alpha}(\chi(\gamma_0)^s - 1)$. Then $H_{\text{Iw}}^2(V, D_{\alpha})$ is \mathcal{H} -torsion and

$$f_{\alpha} \in \text{char } \mathcal{H} H_{\text{Iw}}^2(V, D_{\alpha})$$

(see [49], Theorem 5.4). In the ordinary case the opposite inclusion was recently proved under some technical conditions by Skinner and Urban [52]. It would be interesting to understand whether those method generalises to the non ordinary case.

5) It is certainly possible to formulate the Main Conjecture for families of p -adic representations in the spirit of [29].

0.5. The plan of the paper. The first part of the paper is written as a survey article. In Section 1 we review the theory of (φ, Γ) -modules. In particular, we discuss recent results of Kedlaya, Pottharst and Xiao [39] about the cohomology of (φ, Γ) -modules in families and its applications to the Iwasawa cohomology. In Section 2 we apply this theory to the global Iwasawa theory. The notion of a regular submodule is discussed in Section 2.1. In Sections 2.2-2.4 we construct the complex $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ and review its basic properties following Pottharst [49].

The Main conjecture is formulated in Section 2.5. In the rest of the paper we discuss the relationship between the nullity of $H_{\text{Iw}}^1(V, D)$ and the structure of the semistable module $\mathbf{D}_{\text{st}}(V)$. In Section 3 we consider the weight ≤ -2 case and prove that $H_{\text{Iw}}^1(V, D) = 0$ (and therefore $H_{\text{Iw}}^2(V, D)$ is \mathcal{H} -torsion) if the ℓ -invariant $\ell(V, D)$ constructed in [5], [7] does not vanish. In Section 4 we consider the weight -1 case and prove that the nullity of $H_{\text{Iw}}^1(V, D)$ follows from the non degeneracy of the p -adic height pairing. The proof is based on the generalisation of Nekovář's construction of the p -adic height pairing [43] to the non ordinary case. A systematic study of p -adic heights using (φ, Γ) -modules will be done in [8].

1. (φ, Γ) -MODULES

1.1. (φ, Γ) -modules. In this section we summarize the results about (φ, Γ) -modules which will be used in subsequent Sections. The notion of a (φ, Γ) -module was introduced by Fontaine in the fundamental paper [22]. We consider only (φ, Γ) -modules over \mathbf{Q}_p and those families because this is sufficient for applications we have in mind and refer to original papers for the general case.

Let p be a prime number. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of \mathbf{Q}_p and set $G_{\mathbf{Q}_p} = \text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let $\mathbf{Q}_p^{\text{cyc}} = \mathbf{Q}_p(\mu_{p^\infty})$ and $\Gamma = \text{Gal}(\mathbf{Q}_p^{\text{cyc}}/\mathbf{Q}_p)$. The cyclotomic character $\chi : G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_p^*$ induces an isomorphism of Γ onto \mathbf{Z}_p^* which we denote again by $\chi : \Gamma \rightarrow \mathbf{Z}_p^*$. Let \mathbf{C}_p be the p -adic completion of $\overline{\mathbf{Q}}_p$. We denote by $|\cdot|_p$ the p -adic absolute value on \mathbf{C}_p normalized by $|p|_p = 1/p$. We fix a coefficient field E which will be a finite extension of \mathbf{Q}_p and consider the following objects:

- The field \mathcal{E}_E of power series $f(X) = \sum_{k \in \mathbf{Z}} a_k X^k$, $a_k \in E$ such that a_k

are p -adically bounded and $a_k \rightarrow 0$ when $k \rightarrow -\infty$. Thus \mathcal{E}_E is a complete discrete valuation field with residue field $k_E((X))$ where k_E is the residue field of E .

- For each $0 \leq r < 1$ the ring $\mathcal{R}_E^{(r)}$ of p -adic functions

$$(5) \quad f(X) = \sum_{k \in \mathbf{Z}} a_k X^k, \quad a_k \in E$$

which are holomorphic on the p -adic annulus

$$A(r, 1) = \{z \in \mathbf{C}_p \mid p^{-1/r} \leq |z|_p < 1\}.$$

The Robba ring of power series with coefficients in E is defined as

$$\mathcal{R}_E = \bigcup_r \mathcal{R}_E^{(r)}.$$

Note that \mathcal{R}_E is a Bézout ring (each finitely generated ideal is principal) [40] but it is not noetherian. Its group of units coincides with the group of units of $\mathcal{O}_E[[X]] \otimes \mathbf{Q}_p$ where \mathcal{O}_E is the ring of integers of E .

• For each $0 \leq r < 1$ the ring $\mathcal{E}_E^{\dagger,r}$ of p -adic functions (5) that are bounded on $A(r, 1)$. Then $\mathcal{E}_E^{\dagger} = \bigcup_r \mathcal{E}_E^{\dagger,r}$ is a field which is contained both in \mathcal{E}_E and \mathcal{R}_E . Its elements are called overconvergent power series.

The rings \mathcal{E}_E , \mathcal{E}_E^{\dagger} and \mathcal{R}_E are equipped with an E -linear continuous action of Γ defined by

$$g(f(X)) = f((1+X)^{\chi(g)} - 1), \quad g \in \Gamma$$

and a linear operator φ called the Frobenius and given by

$$\varphi(f(X)) = f((1+X)^p - 1).$$

Note that the actions of Γ and φ commute with each other. Set $t = \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n}$. Then $t \in \mathcal{R}$ and $\gamma(t) = \chi(\gamma)t$, $\varphi(t) = pt$.

Definition. *i) A (φ, Γ) -module over $R = \mathcal{E}_E$, \mathcal{E}_E^{\dagger} or \mathcal{R}_E is a finitely generated free R -module \mathbf{D} equipped with commuting semilinear actions of Γ and φ and such that $A\varphi(\mathbf{D}) = \mathbf{D}$. The last condition means simply that $\varphi(e_1), \dots, \varphi(e_d)$ is a basis of \mathbf{D} if e_1, \dots, e_d is.*

We denote by $\mathbf{M}_A^{\varphi, \Gamma}$ the category of (φ, Γ) -modules over A . The category $\mathbf{M}_{\mathcal{E}_E}^{\varphi, \Gamma}$ contains the important subcategory $\mathbf{M}_{\mathcal{E}_E}^{\varphi, \Gamma, \text{ét}}$ of étale modules. A (φ, Γ) -module \mathbf{D} over \mathcal{E}_E is étale if it is isoclinic of slope 0 in the sense of Dieudonné-Manin's theory. More explicitly, \mathbf{D} is étale if there exists a (φ, Γ) -stable lattice $\mathcal{O}_{\mathcal{E}_E}e_1 + \dots + \mathcal{O}_{\mathcal{E}_E}e_d$ of \mathbf{D} over the ring of integers $\mathcal{O}_{\mathcal{E}_E}$ of \mathcal{E}_E such that the matrix of φ in the basis $\{e_1, \dots, e_d\}$ is invertible over $\mathcal{O}_{\mathcal{E}_E}$. The category $\mathbf{M}_{\mathcal{E}_E^{\dagger}}^{\varphi, \Gamma, \text{ét}}$ of étale modules over \mathcal{E}_E^{\dagger} can be defined by the same manner.

Let $\mathbf{Rep}_E(G_{\mathbf{Q}_p})$ denote the \otimes -category of p -adic representations of the Galois group $G_{\mathbf{Q}_p} = \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ on finite dimensional E -vector spaces. Using the field-of-norms functor of Fontaine-Wintenberger [54] Fontaine constructed an equivalence of \otimes -categories

$$\mathbf{D} : \mathbf{Rep}_E(G_{\mathbf{Q}_p}) \rightarrow \mathbf{M}_{\mathcal{E}_E}^{\varphi, \Gamma, \text{ét}}$$

and conjectured that each p -adic representation V is overconvergent i.e. $\mathbf{D}(V)$ has a canonical \mathcal{E}_E^{\dagger} -lattice $\mathbf{D}^{\dagger}(V)$ stable under the actions of φ and Γ . This was proved by Cherbonnier and Colmez in [14].

The relationship between p -adic representations and (φ, Γ) -modules over the Robba ring can be summarized in the following diagram

$$\begin{array}{ccc} \mathbf{Rep}_E(G_{\mathbf{Q}_p}) & \xrightarrow{\mathbf{D}^\dagger} & \mathbf{M}_{\mathcal{E}_E^\dagger}^{\varphi, \Gamma, \text{ét}} \\ & \searrow \mathbf{D}_{\text{rig}}^\dagger & \downarrow \otimes_{\mathcal{E}_E^\dagger} \mathcal{R}_E \\ & & \mathbf{M}_{\mathcal{R}_E}^{\varphi, \Gamma}. \end{array}$$

A striking fact is that the vertical arrow is a fully faithful functor. This follows from Kedlaya's generalization of Dieudonné-Manin theory [37]. More precisely, the functor $\mathbf{D} \mapsto \mathbf{D} \otimes_{\mathcal{E}_E^\dagger} \mathcal{R}_E$ establishes an equivalence between $\mathbf{M}_{\mathcal{E}_E^\dagger}^{\varphi, \Gamma, \text{ét}}$ and the category of (φ, Γ) -modules over \mathcal{R}_E of slope 0 in the sense of Kedlaya. See [18], Proposition 1.7 for details.

1.2. Cohomology of (φ, Γ) -modules. On the categories of (φ, Γ) -modules one can define cohomology theories whose formal properties are very similar to properties of continuous Galois cohomology of local fields. The main idea is particularly clear if we work with the category of étale modules. Since $\mathbf{M}_{\mathcal{E}_E^\dagger}^{\varphi, \Gamma, \text{ét}}$ is equivalent to $\mathbf{Rep}_E(G_{\mathbf{Q}_p})$ it is possible to compute continuous Galois cohomology $H^*(\mathbf{Q}_p, V)$ in terms of $\mathbf{D}(V)$. In practice, since the category of p -adic representations has not enough of injective objects one should first work modulo p^m -torsion and consider the category of inductive limits of (φ, Γ) -modules over $\mathcal{O}_{\mathcal{E}_E}/p^m$. This leads to the following results [31], [32]. Let $\Delta = \text{Gal}(\mathbf{Q}_p(\mu_p)/\mathbf{Q}_p)$. Then $\Gamma \simeq \Delta \times \Gamma_0$ where Γ_0 is a procyclic p -group. Fix a topological generator γ_0 of Γ_0 and set $\gamma_n = \gamma_1^n$, $n \geq 0$. Let $K_n = \mathbf{Q}_p(\mu_{p^n})^\Delta$, and $K_\infty = \bigcup_{n \geq 0} K_n$. Then $\Gamma_0 = \text{Gal}(K_\infty/\mathbf{Q}_p)$. If M is a topological module equipped with a continuous action of Γ and an operator φ which commute to each other we denote by $C_{\varphi, \gamma_n}^\bullet(M)$ the complex

$$(6) \quad C_{\varphi, \gamma_n}^\bullet(M) : 0 \rightarrow M^\Delta \xrightarrow{d_0} M^\Delta \oplus M^\Delta \xrightarrow{d_1} M^\Delta \rightarrow 0, \quad n \geq 0$$

where $d_0(x) = ((\varphi - 1)x, (\gamma_n - 1)x)$ and $d_1(y, z) = (\gamma_n - 1)y - (\varphi - 1)z$ and define

$$H^*(K_n, M) = H^*(C_{\varphi, \gamma_n}^\bullet(M)).$$

Let \mathbf{D} be an étale (φ, Γ) -module over \mathcal{E}_E and let V be a p -adic representation of $G_{\mathbf{Q}_p}$ such that $\mathbf{D} = \mathbf{D}(V)$. Then there exist isomorphisms

$$H^*(K_n, \mathbf{D}) \simeq H^*(K_n, V)$$

which are functorial and canonical up to the choice of the generator $\gamma \in \Gamma$ (see [31]). This gives an alternative approach to the Euler-Poincaré characteristic formula and the Poincaré duality for Galois cohomology of local fields [31], [32]. If \mathbf{D} is an étale (φ, Γ) -module over \mathcal{E}_E^\dagger , the theorem of Cherbonnier-Colmez implies that again $H^i(K_n, \mathbf{D}) \simeq H^i(K_n, V)$ where V is a p -adic representation such that $\mathbf{D} = \mathbf{D}_{\text{rig}}^\dagger(V)$. The cohomology of (φ, Γ) -modules over \mathcal{R}_E was studied in detail in [41] using a non trivial reduction to the slope 0 case. For any (φ, Γ) -module \mathbf{D} over \mathcal{R}_E we consider $\mathbf{D}(\chi) = \mathbf{D} \otimes_E E(\chi)$ equipped with diagonal actions of Γ and φ (here φ acts trivially on $E(\chi)$). The main properties of the cohomology groups $H^i(K_n, \mathbf{D})$ of \mathbf{D} are:

1) *Long cohomology sequence.* A short exact sequence of (φ, Γ) -modules over \mathcal{R}_E

$$0 \rightarrow \mathbf{D}' \rightarrow \mathbf{D} \rightarrow \mathbf{D}'' \rightarrow 0$$

gives rise to an exact sequence

$$(7) \quad 0 \rightarrow H^0(K_n, \mathbf{D}') \rightarrow H^0(K_n, \mathbf{D}) \rightarrow H^0(K_n, \mathbf{D}) \xrightarrow{\delta^0} H^1(K_n, \mathbf{D}') \rightarrow \dots \rightarrow H^2(K_n, \mathbf{D}'') \rightarrow 0.$$

2) *Euler-Poincaré characteristic.* Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_E . Then $H^i(\mathbf{D})$ are finite dimensional E -vector spaces and

$$(8) \quad \chi(K_n, \mathbf{D}) = \sum_{i=0}^2 (-1)^i \dim_E H^i(K_n, \mathbf{D}) = -[K_n : \mathbf{Q}_p] \text{rank}_{\mathcal{R}_E}(\mathbf{D}).$$

3) *Computation of the Brauer group.* The map

$$(9) \quad \text{cl}(x) \mapsto -\frac{p^n}{\log \chi(\gamma_n)} \text{res}(xdt)$$

is well defined and induces an isomorphism $\text{inv}_{K_n} : H^2(K_n, \mathcal{R}_E(\chi)) \xrightarrow{\sim} E$.

4) *Cup-products.* Let \mathbf{D}' and \mathbf{D}'' be two (φ, Γ) -modules over \mathcal{R}_E . For all i and j such that $i + j \leq 2$ define a bilinear map

$$\cup : H^i(K_n, \mathbf{D}') \times H^j(K_n, \mathbf{D}'') \rightarrow H^{i+j}(K_n, \mathbf{D}' \otimes \mathbf{D}'')$$

by

$$(10) \quad \begin{aligned} \text{cl}(x) \cup \text{cl}(y) &= \text{cl}(x \otimes y) && \text{if } i = j = 0, \\ \text{cl}(x) \cup \text{cl}(y_1, y_2) &= \text{cl}(x \otimes y_1, x \otimes y_2) && \text{if } i = 0, j = 1, \\ \text{cl}(x_1, x_2) \cup \text{cl}(y_1, y_2) &= \text{cl}(x_2 \otimes \gamma_n(y_1) - x_1 \otimes \varphi(y_2)) && \text{if } i = 1, j = 1, \\ \text{cl}(x) \cup \text{cl}(y) &= \text{cl}(x \otimes y) && \text{if } i = 0, j = 2. \end{aligned}$$

These maps commute with connecting homomorphisms in the usual sense.

5) *Duality*. Let $\mathbf{D}^* = \text{Hom}_{\mathcal{R}_E}(\mathbf{D}, \mathcal{R}_E)$. For $i = 0, 1, 2$ the cup product

$$(11) \quad H^i(K_n, \mathbf{D}) \times H^{2-i}(K_n, \mathbf{D}^*(\chi)) \xrightarrow{\cup} H^2(K_n, \mathcal{R}_E(\chi)) \simeq E$$

is a perfect pairing.

6) *Comparison with Galois cohomology*. Let \mathbf{D} be an étale (φ, Γ) -module over \mathcal{O}_E^\dagger . Then the map

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) \rightarrow C_{\varphi, \gamma_n}^\bullet(\mathbf{D} \otimes \mathcal{R}_E)$$

induced by the natural map $\mathbf{D} \rightarrow \mathbf{D} \otimes_{\mathcal{O}_E^\dagger} \mathcal{R}_E$ is a quasi-isomorphism. In particular, if V is a p -adic representation of $G_{\mathbf{Q}_p}$ then

$$H^*(K_n, V) \xrightarrow{\sim} H^*(K_n, \mathbf{D}_{\text{rig}}^\dagger(V)).$$

1.3. Relation to the p -adic Hodge theory. In [22], Fontaine proposed to classify p -adic representations arising in the p -adic Hodge theory in terms of (φ, Γ) -modules (Fontaine's program). More precisely, the problem is to recover classical Fontaine's functors $\mathbf{D}_{\text{dR}}(V)$, $\mathbf{D}_{\text{st}}(V)$ and $\mathbf{D}_{\text{cris}}(V)$ (see for example [25]) from $\mathbf{D}_{\text{rig}}^\dagger(V)$. The complete solution was obtained by Berger in [9], [11]. His theory also allows to prove that each de Rham representation is potentially semistable. See also [17] for introduction and relationship to the theory of p -adic differential equations. In this section we review some of results of Berger in the case where the ground field is \mathbf{Q}_p . Consider the following categories

- The category \mathbf{MF}_E of finite dimensional E vector spaces M equipped with an exhaustive decreasing filtration $(\text{Fil}^i M)_{i \in \mathbf{Z}}$.
- The category $\mathbf{MF}_E^{\varphi, N}$ finite dimensional E vector spaces M equipped with an exhaustive decreasing filtration $(\text{Fil}^i M)_{i \in \mathbf{Z}}$, a linear bijective Frobenius map $\varphi : M \rightarrow M$ and a nilpotent operator (monodromy) $N : M \rightarrow M$ such that $\varphi N = p \varphi N$.
- The subcategory \mathbf{MF}_E^φ of $\mathbf{MF}_E^{\varphi, N}$ formed by filtered (φ, N) -modules M such that $N = 0$ on M .

Let $\mathbf{Q}_p^{\text{cyc}}((t))$ be the ring of Laurent power series equipped with the filtration $\text{Fil}^i \mathbf{Q}_p^{\text{cyc}}((t)) = t^i \mathbf{Q}_p^{\text{cyc}}[[t]]$ and the action of Γ given by $g \left(\sum_{k \in \mathbf{Z}} a_k t^k \right) = \sum_{k \in \mathbf{Z}} g(a_k) \chi(g)^k t^k$. The ring \mathcal{R}_E can not be naturally embedded in $E \otimes \mathbf{Q}_p^{\text{cyc}}((t))$ but for any $r > 0$ small enough and $n \gg 0$ there exists a Γ -equivariant embedding $i_n : \mathcal{R}_E^{(r)} \rightarrow E \otimes \mathbf{Q}_p^{\text{cyc}}((t))$ which sends X onto $\zeta_{p^n} e^{t/p^n} - 1$. Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_E . One can construct for such

r a natural Γ -invariant $\mathcal{R}_E^{(r)}$ -lattice $\mathbf{D}^{(r)}$. Then

$$\mathcal{D}_{\text{dR}}(\mathbf{D}) = \left(E \otimes \mathbf{Q}_p^{\text{cyc}}((t)) \otimes_{i_n} \mathbf{D}^{(r)} \right)^\Gamma$$

is a finite dimensional E -vector space equipped with a decreasing filtration

$$\text{Fil}^i \mathcal{D}_{\text{dR}}(\mathbf{D}) = \left(E \otimes \text{Fil}^i \mathbf{Q}_p^{\text{cyc}}((t)) \otimes_{i_n} \mathbf{D}^{(r)} \right)^\Gamma$$

which does not depend on the choice of r and n .

Let $\mathcal{R}_E[\ell_X]$ denote the ring of power series with coefficients in \mathcal{R}_E . Extend the actions of φ and Γ to $\mathcal{R}_E[\ell_X]$ setting

$$\varphi(\ell_X) = p\ell_X + \log\left(\frac{\varphi(X)}{X^p}\right), \quad g(\ell_X) = \ell_X + \log\left(\frac{g(X)}{X}\right), \quad g \in \Gamma$$

(note that $\log(\varphi(X)/X^p)$ and $\log(g(X)/X)$ converge in \mathcal{R}_E). Define a monodromy operator $N : \mathcal{R}_E[\ell_X] \rightarrow \mathcal{R}_E[\ell_X]$ by $N = -\left(1 - \frac{1}{p}\right)^{-1} \frac{d}{d\log X}$. For any (φ, Γ) -module \mathbf{D} define

$$(12) \quad \mathcal{D}_{\text{st}}(\mathbf{D}) = (\mathbf{D} \otimes_{\mathcal{R}_E} \mathcal{R}_E[\ell_X, 1/t])^\Gamma, \quad t = \log(1+X),$$

$$(13) \quad \mathcal{D}_{\text{cris}}(\mathbf{D}) = \mathcal{D}_{\text{st}}(\mathbf{D})^{N=0} = (\mathbf{D}[1/t])^\Gamma.$$

Then $\mathcal{D}_{\text{st}}(\mathbf{D})$ is a finite dimensional E -vector space equipped with natural actions of φ and N such that $N\varphi = p\varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration induced by the embeddings i_n . We have therefore three functors

$$\mathcal{D}_{\text{dR}} : \mathbf{M}_{\mathcal{R}_E}^{\varphi, \Gamma} \rightarrow \mathbf{MF}_E, \quad \mathcal{D}_{\text{st}} : \mathbf{M}_{\mathcal{R}_E}^{\varphi, \Gamma} \rightarrow \mathbf{MF}_E^{\varphi, N}, \quad \mathcal{D}_{\text{cris}} : \mathbf{M}_{\mathcal{R}_E}^{\varphi, \Gamma} \rightarrow \mathbf{MF}_E^\varphi.$$

Theorem 1 (BERGER). *Let V be a p -adic representation of $G_{\mathbf{Q}_p}$. Then*

$$\mathbf{D}_*(V) \simeq \mathcal{D}_*(V), \quad * \in \{\text{dR}, \text{st}, \text{cris}\}.$$

Proof. See [9]. □

For any (φ, Γ) -module over \mathcal{R}_E one has

$$\dim_E \mathcal{D}_{\text{cris}}(\mathbf{D}) \leq \dim_E \mathcal{D}_{\text{st}}(\mathbf{D}) \leq \dim_E \mathcal{D}_{\text{dR}}(\mathbf{D}) \leq \text{rg}_{\mathcal{R}_E}(\mathbf{D}).$$

One says that \mathbf{D} is de Rham (resp. semistable, resp. crystalline) if $\dim_L \mathcal{D}_{\text{dR}}(\mathbf{D}) = \text{rg}_{\mathcal{R}_E}(\mathbf{D})$ (resp. $\dim_E \mathcal{D}_{\text{st}}(\mathbf{D}) = \text{rg}_{\mathcal{R}_E}(\mathbf{D})$, resp. $\dim_E \mathcal{D}_{\text{cris}}(\mathbf{D}) = \text{rg}_{\mathcal{R}_E}(\mathbf{D})$). Let $\mathbf{M}_{\mathcal{R}_E, \text{pst}}^{\varphi, \Gamma}$ and $\mathbf{M}_{\mathcal{R}_E, \text{cris}}^{\varphi, \Gamma}$ denote the categories of semistable and crystalline (φ, Γ) -modules respectively. Berger proved (see [11]) that the functors

$$(14) \quad \mathcal{D}_{\text{st}} : \mathbf{M}_{\mathcal{R}_E, \text{st}}^{\varphi, \Gamma} \rightarrow \mathbf{MF}_E^{\varphi, N}, \quad \mathcal{D}_{\text{cris}} : \mathbf{M}_{\mathcal{R}_E, \text{cris}}^{\varphi, \Gamma} \rightarrow \mathbf{MF}_E^\varphi$$

are equivalences of \otimes -categories. If \mathbf{D} is de Rham, the jumps of the filtration $\text{Fil}^i \mathcal{D}_{\text{dR}}(\mathbf{D})$ will be called the Hodge-Tate weights of \mathbf{D} .

1.4. Families of (φ, Γ) -modules. In this section we review the theory of (φ, Γ) -modules in families and its relationship to families of p -adic representations following [12], [38], [48] and [39]. Let A be an affinoid algebra over \mathbf{Q}_p . For each $0 \leq r < 1$ the ring $\mathcal{R}_{\mathbf{Q}_p}^{(r)}$ is equipped with a canonical Fréchet topology (see [9]) and we define $\mathcal{R}_A^{(r)} = A \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{R}_{\mathbf{Q}_p}^{(r)}$. Set $\mathcal{R}_A = \bigcup_{0 \leq r < 1} \mathcal{R}_A^{(r)}$. The actions of φ and Γ on $\mathcal{R}_{\mathbf{Q}_p}$ extend by linearity to \mathcal{R}_A . We remark that $\mathcal{R}_A^{(r)}$ is stable under the action of Γ and that $\varphi(\mathcal{R}_A^{(r)}) \subset \mathcal{R}_A^{(r^{1/p})}$ (but $\varphi(\mathcal{R}_A^{(r)}) \not\subset \mathcal{R}_A^{(r)}$). In order to obtain objects with reasonable behavior one defines first (φ, Γ) -modules over $\mathcal{R}_A^{(r)}$. (φ, Γ) -modules over \mathcal{R}_A are defined by extension of coefficients from $\mathcal{R}_A^{(r)}$ to \mathcal{R}_A .

Definition. *i) A (φ, Γ) -module over $\mathcal{R}_A^{(r)}$ is a finite projective $\mathcal{R}_A^{(r)}$ -module $\mathbf{D}^{(r)}$ equipped with the following structures*

a) An isomorphism of $\mathcal{R}_A^{(r^{1/p})}$ -modules

$$\varphi^* : \mathbf{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}, \varphi} \mathcal{R}_A^{(r^{1/p})} \xrightarrow{\sim} \mathbf{D}^{(r^{1/p})}.$$

b) A semilinear continuous action of Γ on $\mathbf{D}^{(r)}$.

ii) One says that \mathbf{D} is a (φ, Γ) -module over \mathcal{R}_A if $\mathbf{D} = \mathbf{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A$ for some (φ, Γ) -module $\mathbf{D}^{(r)}$ over $\mathcal{R}_A^{(r)}$.

The following proposition shows that if $A = E$ this definition is compatible with the definition given in Section 1.1.

Proposition 1. *Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_E . There exists $0 \leq r(\mathbf{D}) < 1$ such that for any r such that $r(\mathbf{D}) \leq r < 1$ there exists a unique free $\mathcal{R}_E^{(r)}$ -submodule $\mathbf{D}^{(r)}$ of \mathbf{D} having the following properties*

a) $\mathbf{D}^{(s)} = \mathbf{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}} \mathcal{R}_E^{(s)}$ for $r \leq s < 1$.

b) $\mathbf{D} = \mathbf{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}} \mathcal{R}_E$

c) The Frobenius φ induces isomorphisms

$$\varphi^* : \mathbf{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}, \varphi} \mathcal{R}_E^{(r^{1/p})} \xrightarrow{\sim} \mathbf{D}^{(r^{1/p})}, \quad r(\mathbf{D}) \leq r < 1.$$

Proof. This is Theorem 1.3.3 of [11]. □

Let $\mathbf{Rep}_A(G_{\mathbf{Q}_p})$ be the category of projective A -modules of finite rank equipped with an A -linear continuous action of $G_{\mathbf{Q}_p}$. The construction of the functor $\mathbf{D}_{\text{rig}}^\dagger$ can be directly generalized to this case [12], [38]. More

precisely, there exists a fully faithful exact functor

$$\mathbf{D}_{\text{rig},A}^\dagger : \mathbf{Rep}_A(G_{\mathbf{Q}_p}) \rightarrow \mathbf{M}_{\mathcal{R}_A}^{\varphi,\Gamma}$$

which commutes with base change. Let $\mathcal{X} = \text{Spm}(A)$. For each $x \in \mathcal{X}$ we denote by \mathfrak{m}_x the maximal ideal of A associated to x and $E_x = A/\mathfrak{m}_x$. If V (resp. \mathbf{D}) is an object of $\mathbf{Rep}_A(G_{\mathbf{Q}_p})$ (resp. of $\mathbf{M}_{\mathcal{R}_A}^{\varphi,\Gamma}$) we set $V_x = V \otimes_A E_x$ (resp. $\mathbf{D}_x = \mathbf{D} \otimes_A E_x$). Then the diagram

$$\begin{array}{ccc} \mathbf{Rep}_A(G_{\mathbf{Q}_p}) & \xrightarrow{\mathbf{D}_{\text{rig},A}^\dagger} & \mathbf{M}_{\mathcal{R}_A}^{\varphi,\Gamma} \\ \downarrow \otimes E_x & & \downarrow \otimes E_x \\ \mathbf{Rep}_{E_x}(G_{\mathbf{Q}_p}) & \xrightarrow{\mathbf{D}_{\text{rig}}^\dagger} & \mathbf{M}_{\mathcal{R}_{E_x}}^{\varphi,\Gamma} \end{array}$$

commutes i.e. $\mathbf{D}_{\text{rig},A}^\dagger(V)_x \simeq \mathbf{D}_{\text{rig}}^\dagger(V_x)$. We remark that in general the essential image of $\mathbf{D}_{\text{rig},A}^\dagger$ does not coincide with the subcategory of étale modules. See [12] [39], [30] for further discussion.

Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_A . As in the case $A = E$ we attach to \mathbf{D} Fontaine-Herr complexes $C_{\varphi, \gamma_n}^\bullet(\mathbf{D})$ and consider the associated cohomology groups $H^*(K_n, \mathbf{D})$. We summarize the main properties of these cohomology in the theorem below. The key result here is the finiteness of the rank of $H^*(K_n, \mathbf{D})$.

Theorem 2. *Let A be an affinoid algebra over \mathbf{Q}_p and let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_A . Then*

i) *Finiteness. The cohomology groups $H^i(K_n, \mathbf{D})$ are finitely generated A -modules. More precisely, for each $n \geq 0$ the complex $C_{\varphi, \gamma_n}^\bullet(\mathbf{D})$ is quasi-isomorphic to the complex of projective A -modules of finite rank concentrated in degrees 0, 1 and 2.*

ii) *Base change. If $f : A \rightarrow B$ is a morphism of affinoid algebras, then*

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) \otimes_{\mathcal{R}_A}^{\mathbf{L}} \mathcal{R}_B \xrightarrow{\sim} C_{\varphi, \gamma_n}^\bullet(\mathbf{D} \widehat{\otimes}_{\mathcal{R}_A} \mathcal{R}_B).$$

In particular, if $x \in \mathcal{X}$, then

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) \otimes_{\mathcal{R}_A}^{\mathbf{L}} E_x \xrightarrow{\sim} C_{\varphi, \gamma_n}^\bullet(\mathbf{D}_x).$$

iii) *Euler-Poincaré formula. One has*

$$\chi(K_n, \mathbf{D}) = \sum_{i=0}^2 \text{rank}_A H^i(K_n, \mathbf{D}) = -[K_n : \mathbf{Q}_p] \text{rg}_{\mathcal{R}_A}(\mathbf{D})$$

where the rank is considered as a function $\text{rg}_A : \text{Spm}(A) \rightarrow \mathbf{N}$.

iv) *Duality. The formulas (10) define a duality*

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) \xrightarrow{\sim} \mathbf{RHom}_A(C_{\varphi, \gamma_n}^\bullet(\mathbf{D}^*(\chi)), A)[-2].$$

v) *Comparison with Galois cohomology.* Let V be a p -adic representation with coefficients in A . Then there are functorial isomorphisms

$$H^*(K_n, V) \xrightarrow{\sim} H^*(K_n, \mathbf{D}_{\text{rig}, A}^\dagger(V))$$

Proof. See [39], Theorems 4.4.1, 4.4.2, 4.4.5 and [48], Theorem 2.8. \square

1.5. Derived categories. There exists the derived version of the comparison isomorphisms v) of Theorem 2 which is important for the formalism of Iwasawa theory [7], [48]. Let A be an affinoid algebra over \mathbf{Q}_p . For any (φ, Γ) -module \mathbf{D} over \mathcal{R}_A define

$$C_{\gamma_n}^\bullet(\mathbf{D}) = \left[\mathbf{D} \xrightarrow{\gamma_n^{-1}} \mathbf{D} \right]$$

where the first term is placed in degree 0. Then $C_{\varphi, \gamma_n}^\bullet(\mathbf{D})$ can be defined as the total complex

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) = \text{Tot}^\bullet \left(C_{\gamma_n}^\bullet(\mathbf{D}) \xrightarrow{\varphi^{-1}} C_{\gamma_n}^\bullet(\mathbf{D}) \right).$$

If \mathbf{D}' and \mathbf{D}'' are two (φ, Γ) -modules over \mathcal{R}_A , the cup product

$$\cup_\gamma : C_{\gamma_n}^\bullet(\mathbf{D}') \otimes C_{\gamma_n}^\bullet(\mathbf{D}'') \rightarrow C_{\gamma_n}^\bullet(\mathbf{D}' \otimes \mathbf{D}'')$$

defined by

$$\cup_\gamma(x_i \otimes y_j) = \begin{cases} x_i \otimes \gamma^n(y_j) & \text{if } x_i \in C_{\gamma_n}^i(\mathbf{D}'), y_j \in C_{\gamma_n}^j(\mathbf{D}'') \text{ and } i+j=0 \text{ or } 1, \\ 0 & \text{if } i+j \geq 2. \end{cases}$$

gives rise to a map of complexes

$$\cup : C_{\varphi, \gamma_n}^\bullet(\mathbf{D}') \otimes C_{\varphi, \gamma_n}^\bullet(\mathbf{D}'') \rightarrow C_{\varphi, \gamma_n}^\bullet(\mathbf{D}' \otimes \mathbf{D}'').$$

Explicitly

$$(15) \quad \cup((x_{i-1}, x_i) \otimes (y_{j-1}, y_j)) = (x_i \cup_\gamma y_{j-1} + (-1)^j x_{i-1} \cup_\gamma \varphi(y_j), x_i \cup_\gamma y_j)$$

if $(x_{i-1}, x_i) \in C_{\varphi, \gamma_n}^i(\mathbf{D}') = C_{\gamma_n}^{i-1}(\mathbf{D}') \oplus C_{\gamma_n}^i(\mathbf{D}')$ and $(y_{j-1}, y_j) \in C_{\varphi, \gamma_n}^j(\mathbf{D}'') = C_{\gamma_n}^{j-1}(\mathbf{D}'') \oplus C_{\gamma_n}^j(\mathbf{D}'')$. It is easy to see that this is a bilinear map which induces the cup-product (11) on cohomology groups.

Let $H_{\mathbf{Q}_p} = \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p^{\text{cyc}})$. In [9], Berger constructed a topological ring $\widetilde{\mathbf{B}}_{\text{rig}, A}^{\dagger, r}$ equipped with commuting actions of φ and $G_{\mathbf{Q}_p}$ such that $\mathbf{D}_{\text{rig}, A}^{\dagger, r}(V) \subset V \otimes_A \widetilde{\mathbf{B}}_{\text{rig}, A}^{\dagger, r}$ for each Galois representation V . Moreover one has an exact sequence

$$(16) \quad 0 \rightarrow A \rightarrow \widetilde{\mathbf{B}}_{\text{rig}, A}^{\dagger, r} \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}_{\text{rig}, A}^{\dagger, r/p} \rightarrow 0$$

(see [10], Lemma 1.7). Set $\widetilde{\mathbf{B}}_{\text{rig},A}^\dagger = \varinjlim_{0 < r} \widetilde{\mathbf{B}}_{\text{rig},A}^{\dagger,r}$. Passing to limits in (16) we obtain an exact sequence

$$(17) \quad 0 \rightarrow A \rightarrow \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger \rightarrow 0$$

Set $G_{K_n} = \text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\mu_{p^n}))$. Tensoring (17) with a p -adic representation V and taking continuous cochains one obtains an exact sequence

$$0 \rightarrow C^\bullet(G_{K_n}, V) \rightarrow C^\bullet(G_{K_n}, V \otimes \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger) \xrightarrow{\varphi-1} C^\bullet(G_{K_n}, V \otimes \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger) \rightarrow 0.$$

Define

$$(18) \quad K^\bullet(K_n, V) = \text{Tot}^\bullet \left(C^\bullet \left(G_{K_n}, V \otimes \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger \right) \xrightarrow{\varphi-1} C^\bullet \left(G_{K_n}, V \otimes \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger \right) \right).$$

Consider the diagram

$$(19) \quad \begin{array}{ccc} C^\bullet(G_{K_n}, V) & \xrightarrow{\beta_V} & K^\bullet(K_n, V) \\ & \uparrow \alpha_V & \\ & C_{\varphi, \gamma_n}^\bullet(V) & \end{array}$$

where the maps α_V and β_V are constructed as follows. Let $\alpha_{V,\gamma} : C_{\gamma_n}^\bullet(V) \rightarrow C^\bullet(G_{\mathbf{Q}_p}, V \otimes \widetilde{\mathbf{B}}_{\text{rig},A}^\dagger)$ be the morphism defined by

$$\begin{aligned} \alpha_{V,\gamma}(x_0) &= x_0, & \text{if } x_0 \in C_{\gamma_n}^0(V), \\ \alpha_{V,\gamma}(x_1)(g) &= \frac{g-1}{\gamma_n-1}(x_1), & \text{if } x_1 \in C_{\gamma_n}^1(V). \end{aligned}$$

Then α_V is the map induced by $\alpha_{V,\gamma}$ by passing to total complexes. The map β_V is defined by

$$\begin{aligned} \beta_V : C^\bullet(G_{K_n}, V) &\rightarrow K^\bullet(K_n, V), \\ x_n &\mapsto (0, x_n), \quad x_n \in C^n(G_{K_n}, V). \end{aligned}$$

Let $\mathbf{R}\Gamma(K_n, V)$ and $\mathbf{R}\Gamma(K_n, \mathbf{D}_{\text{rig}}^\dagger(V))$ denote the images of complexes $C^\bullet(G_{K_n}, V)$ and $C_{\varphi, \gamma_n}^\bullet(\mathbf{D}_{\text{rig}}^\dagger(V))$ in the derived category $\mathbf{D}^b(A)$ of A -modules.

Proposition 2. *i) In the diagram (19) the maps α_V and β_V are quasi isomorphisms and therefore*

$$(20) \quad \mathbf{R}\Gamma(K_n, V) \xrightarrow{\sim} \mathbf{R}\Gamma(K_n, \mathbf{D}_{\text{rig}}^\dagger(V))$$

in $\mathbf{D}^b(A)$.

Proof. See [7], Proposition A.1. □

1.6. Iwasawa cohomology. Let \mathcal{O}_E denote the ring of integers of the field of coefficients E . We equip the Iwasawa algebra $\Lambda = \mathcal{O}_E[[\Gamma_1]]$ with the involution $\iota : \Lambda \rightarrow \Lambda$ defined by $\iota(g) = g^{-1}$, $g \in \Gamma_1$. Let V be a p -adic representation of $G_{\mathbf{Q}_p}$ with coefficients in E . Fix a \mathcal{O}_E -lattice T of V stable under the action of $G_{\mathbf{Q}_p}$. The induced module $\text{Ind}_{K_\infty/\mathbf{Q}_p}(T) = \Lambda \otimes_{\mathcal{O}_E} T$ is equipped with the diagonal action of $G_{\mathbf{Q}_p}$ and the natural structure of a Λ -module given by $\lambda * m = \iota(\lambda)m$ (see for example [43], Chapter 8. To fix these structures we will write $\text{Ind}_{K_\infty/\mathbf{Q}_p}(T) = (\Lambda \otimes_{\mathbf{Z}_p} T)^\iota$. Let $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_p, T)$ denote the class of the complex $C^\bullet(G_{\mathbf{Q}_p}, \text{Ind}_{K_\infty/\mathbf{Q}_p}(T))$ in the derived category $D^b(\Lambda)$. Define

$$H_{\text{Iw}}^i(\mathbf{Q}_p, T) = \mathbf{R}^i\Gamma_{\text{Iw}}(\mathbf{Q}_p, T), \quad i \in \mathbf{N}.$$

From Shapiro's lemma it follows that there are canonical and functorial isomorphisms in $D^b(\mathbf{Z}_p[G_n])$ where $G_n = \text{Gal}(K_n/\mathbf{Q}_p)$

$$H_{\text{Iw}}^i(\mathbf{Q}_p, T) \simeq \varprojlim_{\text{cores}_{K_n/K_{n-1}}} H^i(K_n, T),$$

$$\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_p, T) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{Z}_p[G_n] \simeq \mathbf{R}\Gamma(K_n, T).$$

We review the computation of Iwasawa cohomology in terms of (φ, Γ) -modules. It was found by Fontaine (unpublished but see [15]). Let

$$\psi : \mathcal{O}_E^\dagger \rightarrow \mathcal{O}_E^\dagger$$

denote the operator

$$\psi(f(X)) = \frac{1}{p} \text{Tr}_{\mathcal{O}_E^\dagger/\varphi(\mathcal{O}_E^\dagger)}(f(X)).$$

More explicitly, the polynomials $1, (1+X), \dots, (1+X)^{p-1}$ form a basis of \mathcal{O}_E^\dagger over $\varphi(\mathcal{O}_E^\dagger)$ and one has

$$\psi\left(\sum_{i=0}^{p-1} \varphi(f_i)(1+X)^i\right) = f_0.$$

In particular, $\psi \circ \varphi = \text{id}$. Let e_1, \dots, e_d be a base of $\mathbf{D}^\dagger(V)$ over \mathcal{O}_E^\dagger . Then $\varphi(e_1), \dots, \varphi(e_d)$ is again a base of $\mathbf{D}^\dagger(V)$ and we define

$$\psi : \mathbf{D}^\dagger(V) \rightarrow \mathbf{D}^\dagger(V)$$

by $\psi\left(\sum_{i=1}^d a_i \varphi(e_i)\right) = \sum_{i=1}^d \psi(a_i) e_i$. We remark that this definition extends to (φ, Γ) -modules over the Robba ring.

Consider the complex

$$C_{\text{Iw}}^\bullet(\mathbf{D}^\dagger(T)) : \mathbf{D}^\dagger(T)^\Delta \xrightarrow{\psi-1} \mathbf{D}^\dagger(T)^\Delta$$

where the first term is placed in degree 0 and denote by $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}_{\text{rig}}^\dagger(T))$ the corresponding object in $\mathbf{D}^b(\Lambda)$.

Theorem 3. *The complexes $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_p, T)$, $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}_{\text{rig}}^\dagger(T))$ and $\mathbf{R}\Gamma(\mathbf{Q}_p, \mathbf{D}^\dagger(\text{Ind}_{K_\infty/\mathbf{Q}_p} T))$ are isomorphic in $\mathbf{D}^b(\Lambda)$.*

Proof. See [15] and [7] for the derived version. \square

The analog of previous results for (φ, Γ) -modules over the Robba ring was obtained by Pottharst [49]. We start with some preliminary results about coadmissible \mathcal{H} -modules and the Grothendieck duality. Let $B(0, 1) = \{z \in \mathbf{C}_p \mid |z|_p < 1\}$ denote the open unit disc. Define

$$\mathcal{H} = \{f(X) \in E[[X]] \mid f(X) \text{ converges on } B(0, 1)\}.$$

In the context of Iwasawa theory the ring \mathcal{H} appeared in [46]. The natural description \mathcal{H} is as follows. Write

$$B(0, 1) = \bigcup_n W_n$$

where $W_n = \{z \in \mathbf{C}_p \mid |z|_p \leq p^{-1/n}\}$ is the closed disc of radii $p^{-1/n} < 1$ which we consider as an affinoid space. Then

$$\Gamma(W_n, \mathcal{O}_{W_n}) = \left\{ f(X) = \sum_{k=0}^{\infty} a_k X^k \mid |a_k|_p p^{-k/n} \rightarrow 0 \text{ when } k \rightarrow +\infty \right\}$$

and $\mathcal{H} = \varprojlim_n \Gamma(W_n, \mathcal{O}_{W_n})$. We consider Λ and $\Lambda \otimes_{\mathcal{O}_E} E$ as subalgebras of \mathcal{H} . To simplify notation set $\mathcal{H}_n = \Gamma(W_n, \mathcal{O}_{W_n})$. It is easy to see that \mathcal{H}_n are euclidian rings. A coadmissible \mathcal{H} -module M is the inverse limit of a system $(M_n)_{n \geq 1}$ where each M_n is a finitely generated \mathcal{H}_n -module and the natural maps $M_n \otimes_{\mathcal{H}_n} \mathcal{H}_{n-1} \rightarrow M_{n-1}$ are isomorphisms [51]. The structure of admissible modules is given by the following proposition ([49], Proposition 1.1).

Proposition 3. *i) A coadmissible \mathcal{H} -module is torsion free if and only if it is a finitely generated \mathcal{H} -module.*

ii) Let M be a coadmissible torsion \mathcal{H} -module. Then

$$(21) \quad M \simeq \prod_{i \in I} \mathcal{H} / \mathfrak{p}_i^{n_i}$$

where $(\mathfrak{p}_i)_{i \in I} \subset \bigcup_{n \geq 1} \text{Spec}(\mathcal{H}_n)$ is a system of maximal ideals such that for each n there are only finitely many i with $\mathfrak{p}_i \in \text{Spec}(\mathcal{H}_n)$.

Proof. The proof follows easily from the theory of Lazard [40]. \square

Let $D_{\text{coad}}^b(\mathcal{H})$ denote the category of bounded complexes of \mathcal{H} -modules with coadmissible cohomology. Let \mathcal{K} denote the field of fractions of \mathcal{H} . Consider the complex

$$\omega = \text{cone}[\mathcal{K} \rightarrow \mathcal{K}/\mathcal{H}][-1]$$

and for any object C^\bullet of $D_{\text{coad}}^b(\mathcal{H})$ define

$$\mathcal{D}(C^\bullet) = \text{Hom}_{\mathcal{H}}(C^\bullet, \omega).$$

Then $\mathcal{D} : D_{\text{coad}}^b(\mathcal{H}) \rightarrow D_{\text{coad}}^b(\mathcal{H})$ is an anti involution which can be seen as the "limit" of Grothendieck dualizing functors $\mathbf{R}\text{Hom}(-, \mathcal{H}_n)$. For any coadmissible module M let $\mathcal{D}^k(M)$ denote the k th cohomology group of $\mathcal{D}([M])$. Then

$$\mathcal{D}^0(M) = \text{Hom}_{\mathcal{H}}(M/M_{\text{tor}}, \mathcal{H}), \quad \mathcal{D}^1\left(\prod_{i \in I} \mathcal{H}/\mathfrak{p}_i^{n_i}\right) = \prod_{i \in I} \mathfrak{p}_i^{-n_i}/\mathcal{H},$$

and $\mathcal{D}^k(M) = 0$ for all $k \geq 2$ (see [49], Section 1).

Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_E . Consider the complexes

$$C_{\text{Iw}}^\bullet(\mathbf{D}) : \mathbf{D}^\Delta \xrightarrow{\psi-1} \mathbf{D}^\Delta$$

and $C_{\varphi, \gamma}^\bullet(\overline{\mathbf{D}})$ where $\overline{\mathbf{D}} = \mathbf{D} \otimes_{\mathbf{Q}_p} \mathcal{H}^1$ and denote by $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D})$ and $\mathbf{R}\Gamma(\mathbf{Q}_p, \overline{\mathbf{D}})$ the corresponding objects of the derived category $D_{\text{coad}}^b(\mathcal{H})$.

Theorem 4. *Let \mathbf{D} be a (φ, Γ) -module over \mathcal{R}_E . Then*

i) $H_{\text{Iw}}^i(\mathbf{D})$ ($i = 1, 2$) are coadmissible \mathcal{H} -modules. Moreover $H_{\text{Iw}}^1(\mathbf{D})_{\text{tor}}$ and $H_{\text{Iw}}^2(\mathbf{D})$ are finite dimensional E -vector spaces.

ii) The complexes $C_{\text{Iw}}^\bullet(\mathbf{D})$ and $C_{\varphi, \gamma}^\bullet(\overline{\mathbf{D}})$ are quasi isomorphic and therefore

$$\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}) \simeq \mathbf{R}\Gamma(\mathbf{Q}_p, \overline{\mathbf{D}}).$$

iii) One has an isomorphism

$$C_{\varphi, \gamma}^\bullet(\overline{\mathbf{D}}) \otimes_{\mathcal{H}} E \xrightarrow{\sim} C_{\varphi, \gamma}^\bullet(\mathbf{D})$$

which induces Hochschild-Serre exact sequences

$$0 \rightarrow H_{\text{Iw}}^i(\mathbf{D})_\Gamma \rightarrow H^i(\mathbf{D}) \rightarrow H_{\text{Iw}}^{i+1}(\mathbf{D})^\Gamma \rightarrow 0.$$

iv) One has a canonical duality in $D_{\text{coad}}^b(\mathcal{H})$

$$\mathcal{D}\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}) \simeq \mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}^*(\chi))^1[2].$$

iv) Let V be a p -adic representation of $G_{\mathbf{Q}_p}$. There exist canonical and functorial isomorphisms

$$\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_p, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \simeq \mathbf{R}\Gamma(\mathbf{Q}_p, T \otimes_{\mathbf{Z}_p} \mathcal{H}^1) \simeq \mathbf{R}\Gamma(\mathbf{Q}_p, \overline{\mathbf{D}}).$$

Proof. See [49], Theorem 2.6. □

1.7. Crystalline extensions. In this subsection we consider (φ, Γ) -modules over the Robba ring \mathcal{R}_E . As usual, the first cohomology group $H^1(\mathbf{Q}_p, \mathbf{D})$ can be interpreted in terms of extensions. Namely, to any cocycle $\alpha = (a, b) \in Z^1(C_{\varphi, \gamma}(\mathbf{D}))$ one associates the extension

$$0 \rightarrow \mathbf{D} \rightarrow \mathbf{D}_\alpha \rightarrow \mathcal{R}_E \rightarrow 0$$

such that $\mathbf{D}_\alpha = \mathbf{D} \oplus \mathcal{R}_E e$ with $\varphi(e) = e + a$ and $\gamma(e) = e + b$. This defines a canonical isomorphism

$$H^1(\mathbf{Q}_p, \mathbf{D}) \simeq \text{Ext}^1(\mathcal{R}_E, \mathbf{D}).$$

We say that $\text{cl}(\alpha) \in H^1(\mathbf{Q}_p, \mathbf{D})$ is crystalline if

$$\dim_E \mathcal{D}_{\text{cris}}(\mathbf{D}_\alpha) = \dim_E \mathcal{D}_{\text{cris}}(\mathbf{D}) + 1$$

and define

$$H_f^1(\mathbf{Q}_p, \mathbf{D}) = \{\text{cl}(\alpha) \in H^1(\mathbf{Q}_p, \mathbf{D}) \mid \text{cl}(\alpha) \text{ is crystalline}\}.$$

It is easy to see that $H_f^1(\mathbf{Q}_p, \mathbf{D})$ is a subspace of $H^1(\mathbf{Q}_p, \mathbf{D})$. If \mathbf{D} is semistable (even potentially semistable), the equivalence (14) between the category of semistable (φ, Γ) -modules and filtered (φ, N) -modules allows to compute $H_f^1(\mathbf{Q}_p, \mathbf{D})$ in terms of $\mathcal{D}_{\text{st}}(\mathbf{D})$. This gives a canonical exact sequence

$$(22) \quad 0 \rightarrow H^0(\mathbf{Q}_p, \mathbf{D}) \rightarrow \mathcal{D}_{\text{cris}}(\mathbf{D}) \rightarrow \mathcal{D}_{\text{cris}}(\mathbf{D}) \oplus t_{\mathbf{D}}(\mathbf{Q}_p) \rightarrow H_f^1(\mathbf{Q}_p, \mathbf{D}) \rightarrow 0,$$

where $t_{\mathbf{D}}(\mathbf{Q}_p) = \mathcal{D}_{\text{st}}(\mathbf{D}) / \text{Fil}^0 \mathcal{D}_{\text{st}}(\mathbf{D})$ is the tangent space of \mathbf{D} ([5], Proposition 1.4.4 and [42], Sections 1.19-1.21). In particular, one has

$$(23) \quad \begin{aligned} H^0(\mathbf{Q}_p, \mathbf{D}) &= \text{Fil}^0 \mathcal{D}_{\text{st}}(\mathbf{D})^{\varphi=1, N=0}, \\ \dim_E H_f^1(\mathbf{Q}_p, \mathbf{D}) &= \dim_E t_{\mathbf{D}}(\mathbf{Q}_p) + \dim_L H^0(\mathbf{Q}_p, \mathbf{D}) \end{aligned}$$

(see [5], Proposition 1.4.4 and Corollary 1.4.5). Moreover, $H_f^1(\mathbf{Q}_p, \mathbf{D})$ and $H_f^1(\mathbf{Q}_p, \mathbf{D}^*(\chi))$ are orthogonal complements to each other under duality (11) ([5], Corollary 1.4.10).

We will call exponential map the connecting map in (22)

$$(24) \quad \exp_{\mathbf{D}} : t_{\mathbf{D}}(\mathbf{Q}_p) \rightarrow H_f^1(\mathbf{Q}_p, \mathbf{D}).$$

Let V be a potentially semistable representation of $G_{\mathbf{Q}_p}$. Let

$$t_V(\mathbf{Q}_p) = \mathbf{D}_{\text{st}}(V) / \text{Fil}^0 \mathbf{D}_{\text{st}}(V)$$

denote the tangent space of V and let $H_f^1(\mathbf{Q}_p, V)$ be the subgroup of $H^1(\mathbf{Q}_p, V)$ defined by (2). In [13], Bloch and Kato constructed a map

$$(25) \quad \exp_V : t_V(\mathbf{Q}_p) \rightarrow H_f^1(\mathbf{Q}_p, V)$$

using the fundamental exact sequence relating the rings of p -adic periods \mathbf{B}_{cris} and \mathbf{B}_{dR} . From Theorem 1 it follows that $t_V(\mathbf{Q}_p)$ is canonically isomorphic to $t_{\mathbf{D}_{\text{rig}}^\dagger(V)}(\mathbf{Q}_p)$ and that

$$H_f^1(\mathbf{Q}_p, \mathbf{D}_{\text{rig}}^\dagger(V)) \simeq H_f^1(\mathbf{Q}_p, V)$$

(see [5], Proposition 1.4.2). Moreover the diagram

$$\begin{array}{ccc} t_V(\mathbf{Q}_p) & \xrightarrow{\exp_V} & H_f^1(\mathbf{Q}_p, V) \\ \downarrow = & & \downarrow = \\ t_{\mathbf{D}_{\text{rig}}^\dagger(V)}(\mathbf{Q}_p) & \xrightarrow{\exp_{\mathbf{D}_{\text{rig}}^\dagger(V)}} & H_f^1(\mathbf{Q}_p, \mathbf{D}_{\text{rig}}^\dagger(V)). \end{array}$$

commutes ([7], Section 2). Therefore, our definition of the exponential map (24) agrees with the Bloch–Kato’s one.

1.8. Cohomology of isoclinic modules. The results of this section are proved in [5] (see Proposition 1.5.9 and Section 1.5.10 of *op. cit.*). Let \mathbf{D} be semistable (φ, Γ) -module over \mathcal{R}_E of rank d . Assume that $\mathcal{D}_{\text{st}}(\mathbf{D})^{\varphi=1} = \mathcal{D}_{\text{st}}(\mathbf{D})$ and that the all Hodge–Tate weights of \mathbf{D} are ≥ 0 . Since $N\varphi = p\varphi N$ this implies that $N = 0$ on $\mathcal{D}_{\text{st}}(\mathbf{D})$ and \mathbf{D} is crystalline.

The canonical map $\mathbf{D}^\Gamma \rightarrow \mathcal{D}_{\text{cris}}(\mathbf{D})$ is an isomorphism and therefore

$$H^0(\mathbf{Q}_p, \mathbf{D}) \simeq \mathcal{D}_{\text{cris}}(\mathbf{D}) = \mathbf{D}^\Gamma$$

is a E -vector space of dimension d . The Euler–Poincaré characteristic formula gives

$$\dim_E H^1(\mathbf{Q}_p, \mathbf{D}) = d + \dim_E H^0(\mathbf{Q}_p, \mathbf{D}) + \dim_E H^0(\mathbf{Q}_p, \mathbf{D}^*(\chi)) = 2d.$$

On the other hand $\dim_E H_f^1(\mathbf{Q}_p, \mathbf{D}) = d$ by (23). The group $H^1(\mathbf{Q}_p, \mathbf{D})$ has the following explicit description. The map

$$\begin{aligned} i_{\mathbf{D}} : \mathcal{D}_{\text{cris}}(\mathbf{D}) \oplus \mathcal{D}_{\text{cris}}(\mathbf{D}) &\rightarrow H^1(\mathbf{Q}_p, \mathbf{D}), \\ i_{\mathbf{D}}(x, y) &= \text{cl}(-x, \log \chi(\gamma) y) \end{aligned}$$

is an isomorphism. (Remark that the sign -1 and $\log \chi(\gamma)$ are normalizing factors.) We let denote $i_{\mathbf{D},f}$ and $i_{\mathbf{D},c}$ the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. Then $\text{Im}(i_{\mathbf{D},f}) = H_f^1(\mathbf{Q}_p, \mathbf{D})$ and we set $H_c^1(\mathbf{Q}_p, \mathbf{D}) = \text{Im}(i_{\mathbf{D},c})$. Thus we have a canonical decomposition

$$(26) \quad H^1(\mathbf{Q}_p, \mathbf{D}) \simeq H_f^1(\mathbf{Q}_p, \mathbf{D}) \oplus H_c^1(\mathbf{Q}_p, \mathbf{D}).$$

Now consider the dual module $\mathbf{D}^*(\chi)$. It is crystalline, $\mathcal{D}_{\text{cris}}(\mathbf{D}^*(\chi))^{\varphi=p^{-1}} = \mathcal{D}_{\text{cris}}(\mathbf{D}^*(\chi))$ and the Hodge–Tate weights of $\mathbf{D}^*(\chi)$

are all ≤ 0 . Let

$$[\ , \]_{\mathbf{D}} : \mathcal{D}_{\text{cris}}(\mathbf{D}^*(\chi)) \times \mathcal{D}_{\text{cris}}(\mathbf{D}) \rightarrow E$$

denote the canonical pairing. Define

$$i_{\mathbf{D}^*(\chi)} : \mathcal{D}_{\text{cris}}(\mathbf{D}^*(\chi)) \oplus \mathcal{D}_{\text{cris}}(\mathbf{D}^*(\chi)) \rightarrow H^1(\mathbf{Q}_p, \mathbf{D}^*(\chi))$$

by

$$i_{\mathbf{D}^*(\chi)}(\alpha, \beta) \cup i_{\mathbf{D}}(x, y) = [\beta, x]_{\mathbf{D}} - [\alpha, y]_{\mathbf{D}}.$$

As before, let $i_{\mathbf{D}^*(\chi), f}$ and $i_{\mathbf{D}^*(\chi), c}$ denote the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. From $H_f^1(\mathbf{Q}_p, \mathbf{D}^*(\chi)) = H_f^1(\mathbf{D})^\perp$ it follows that $\text{Im}(i_{\mathbf{D}^*(\chi), f}) = H_f^1(\mathbf{Q}_p, \mathbf{D}^*(\chi))$ and we set $H_c^1(\mathbf{Q}_p, \mathbf{D}^*(\chi)) = \text{Im}(i_{\mathbf{D}^*(\chi), c})$. Again we have a decomposition

$$H^1(\mathbf{Q}_p, \mathbf{D}^*(\chi)) \simeq H_f^1(\mathbf{Q}_p, \mathbf{D}^*(\chi)) \oplus H_c^1(\mathbf{Q}_p, \mathbf{D}^*(\chi)).$$

Now we will compute the Iwasawa cohomology of some isoclinic modules. Recall that the well known computation of universal norms of $\mathbf{Q}_p(1)$ gives:

$$(27) \quad H_{\text{Iw}}^1(\mathbf{Q}_p, \mathbf{Q}_p(1))_\Gamma = p^{\mathbf{Z}}$$

under the Kummer map. The following proposition generalises this result to (φ, Γ) -modules. It will be used in the proof of Proposition 9 below.

Proposition 4. *Let \mathbf{D} be an isoclinic (φ, Γ) -module such that $\mathcal{D}_{\text{cris}}(\mathbf{D})^{\varphi=p^{-1}} = \mathcal{D}_{\text{cris}}(\mathbf{D})$ and $\text{Fil}^0 \mathcal{D}_{\text{cris}}(\mathbf{D}) = 0$. Then*

$$H_{\text{Iw}}^1(\mathbf{D})_\Gamma = H_c^1(\mathbf{Q}_p, \mathbf{D}).$$

Proof. For any continuous character $\delta : \mathbf{Q}_p^* \rightarrow E^*$ let $\mathcal{R}_E(\delta)$ denote the (φ, Γ) -module of rank one $\mathcal{R}_E e_\delta$ such that $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$. In [18], Proposition 3.1 Colmez proved that any (φ, Γ) -module of rank 1 is isomorphic to a unique module of the form $\mathcal{R}_E(\delta)$ for some δ .

Set $\mathbf{D}_m = \mathcal{R}_E(|x|x^m)$. Let e_m denote the canonical generator of \mathbf{D}_m . It is not difficult to see that an isoclinic (φ, Γ) -module \mathbf{D} which satisfies the conditions of Proposition 4 is isomorphic to the direct sum $\bigoplus_{i=1}^d \mathbf{D}_{m_i}$ for some $m_i \geq 1$ ([5], Proposition 1.5.8). Therefore we can assume that $\mathbf{D} = \mathbf{D}_m$ for some $m \geq 1$. The E -vector space $H^1(\mathbf{D}_m)$ is generated by the cohomology classes $\text{cl}(\alpha_m)$ and $\text{cl}(\beta_m)$ where

$$\begin{aligned} \alpha_m &= \partial^{m-1} \left(\frac{1}{X} + \frac{1}{2}, a \right) e_m, & (1 - \varphi)a &= (1 - \chi(\gamma)\gamma) \left(\frac{1}{X} + \frac{1}{2} \right) \\ \beta_m &= \partial^{m-1} \left(b, \frac{1}{X} \right) e_m, & (1 - \varphi) \left(\frac{1}{X} \right) &= (1 - \chi(\gamma)\gamma)b \end{aligned}$$

where $\partial = (1+X)d/dX$ ([18], Sections 2.3-2.5). Moreover $\text{cl}(\alpha_m) \in H_f^1(\mathbf{D}_m)$ and $\text{cl}(\alpha_m) \in H_c^1(\mathbf{D}_m)$ ([5], Theorem 1.5.7). For $m = 1$ the module $\mathbf{D}_1 = \mathcal{R}_E(\chi)$ is étale and corresponds to the p -adic representation $E(1)$. Using the formula

$$\frac{1}{X} = \sum_{i=0}^{p-1} (1+X)^i \phi\left(\frac{1}{X}\right)$$

it can be checked directly that $\psi(1/X) = 1/X$. Thus $\psi(\frac{1}{X}e_1) = \frac{1}{X}e_1$ and we proved that $\frac{1}{X}e_1 \in \mathbf{D}_1^{\psi=1}$. An easy induction using the identity $\psi\partial = p\partial\psi$ shows that for each $m \geq 1$ one has $\partial^{m-1}(\frac{1}{X})e_m \in \mathbf{D}_m^{\psi=1}$. Since the image of $\partial^{m-1}(\frac{1}{X})e_m$ under the map $H_{\text{Iw}}^1(\mathbf{D}_m) \rightarrow H_{\text{Iw}}^1(\mathbf{D}_m)_\Gamma \subset H^1(\mathbf{D}_m)$ is β_m , this proves that $H_{\text{Iw}}^1(\mathbf{D}_m)_\Gamma \subset H_c^1(\mathbf{D}_m)$. On the other hand, Theorem 4 iii) gives Hochschild-Serre exact sequence

$$0 \rightarrow H_{\text{Iw}}^1(\mathbf{D}_m)_\Gamma \rightarrow H^1(\mathbf{D}_m) \rightarrow H_{\text{Iw}}^2(\mathbf{D}_m)^\Gamma \rightarrow 0.$$

By the same Theorem $H_{\text{Iw}}^2(\mathbf{D}_m)$ is finite dimensional over E and therefore

$$\dim_E H_{\text{Iw}}^2(\mathbf{D}_m)^\Gamma = \dim_E H^1(\mathbf{D}_m)_\Gamma = \dim_E H^2(\mathbf{D}_m) = 1.$$

Thus

$$\dim_E H_{\text{Iw}}^1(\mathbf{D}_m)_\Gamma = \dim_E H^1(\mathbf{D}_m) - \dim_E H^2(\mathbf{D}_m) = 1$$

and we proved that $\dim_E H_{\text{Iw}}^1(\mathbf{D}_m)_\Gamma = \dim_E H_c^1(\mathbf{D}_m)$. This gives the proposition. \square

Remark. It can be shown that the image of $p \in \mathbf{Q}_p^*$ under the Kummer map is $(1 - 1/p) \log(\chi) \text{cl}(\beta_1)$ (see [4], Proposition 2.1.5).

2. THE MAIN CONJECTURE

2.1. Regular submodules. In this section we apply the theory of (ϕ, Γ) -modules to Iwasawa theory of p -adic representations. In particular, we propose a conjecture which can be seen as a (weak) generalisation of both Greenberg's [27] and Perrin-Riou's [47] Main Conjectures. Fix a prime number p and a finite set S of primes of \mathbf{Q} containing p . For each number field F we denote by $G_{F,S}$ the Galois group of the maximal algebraic extension of F unramified outside $S \cup \{\infty\}$. For any topological module M equipped with a continuous action of $G_{F,S}$ we write $H_S^*(F, M)$ for the continuous cohomology of $G_{F,S}$ with coefficients in M . Fix a finite extension E of \mathbf{Q}_p which will play the role of the coefficient field. Let V be a p -adic representation of $G_{\mathbf{Q},S}$ with coefficients in E . Recall that for each $v \in S$ we denote by $H_f^1(\mathbf{Q}_v, V)$ the subgroup of $H^1(\mathbf{Q}_v, V)$ defined by (2) and by $H_f^1(\mathbf{Q}, V)$ the Bloch–Kato Selmer group (1).

The Poitou–Tate exact sequence gives an exact sequence

$$(28) \quad 0 \rightarrow H_f^1(\mathbf{Q}, V) \rightarrow H_S^1(\mathbf{Q}, V) \rightarrow \bigoplus_{v \in S} \frac{H^1(\mathbf{Q}_v, V)}{H_f^1(\mathbf{Q}_v, V)} \rightarrow H_f^1(\mathbf{Q}, V^*(1))^* \rightarrow \\ H_S^2(\mathbf{Q}, V) \rightarrow \bigoplus_{v \in S} H^2(\mathbf{Q}_v, V) \rightarrow H_S^0(\mathbf{Q}, V^*(1))^* \rightarrow 0$$

(see [26], Proposition 2.2.1) Together with the well known formula for the Euler characteristic this implies that

$$(29) \quad \dim_E H_f^1(\mathbf{Q}, V) - \dim_E H_f^1(\mathbf{Q}, V^*(1)) - \dim_E H_S^0(\mathbf{Q}, V) + \\ + \dim_E H_S^0(\mathbf{Q}, V^*(1)) = \dim_E t_V(\mathbf{Q}_p) - d_+(V).$$

where $d_{\pm}(V) = \dim_E V^{c=\pm 1}$ and c denotes the complex conjugation.

In the rest of this section we assume that V satisfies the following conditions:

C1) $H_S^0(\mathbf{Q}, V) = H_S^0(\mathbf{Q}, V^*(1)) = 0$.

C2) V is semistable at p and $\varphi : \mathbf{D}_{\text{st}}(V) \rightarrow \mathbf{D}_{\text{st}}(V)$ is semisimple at 1 and p^{-1} .

C3) $\mathbf{D}_{\text{cris}}(V)^{\varphi=1} = 0$.

C4) V satisfies one of the following conditions

a) $H_f^1(\mathbf{Q}, V^*(1)) = 0$.

b) V is a self dual representation, i.e. it is equipped with a non degenerate bilinear form $V \times V \rightarrow E(1)$.

In support of these assumptions we remark that representations which we have in mind appear as p -adic realisations of pure motives over \mathbf{Q} . Let X/\mathbf{Q} be a smooth proper variety having good reduction outside a finite set S of places containing p . Consider the motives $h^i(X)(m)$ where $0 \leq i \leq 2 \dim(X)$ and $m \in \mathbf{Z}$. Let $H_p^i(X)$ denote the p -adic étale cohomology of X . The p -adic realization of $h^i(X)(m)$ is $V = H_p^i(X)(m)$. The action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ factors through $G_{\mathbf{Q}, S}$. Moreover the restriction of $H_p^i(X)$ on the decomposition group at p is semistable if X has semistable reduction at p (and potentially semistable in general) [20], [53]. Poincaré duality and the hard Lefschetz theorem give a canonical isomorphism

$$(30) \quad H_p^i(X)^* \simeq H_p^i(X)(i)$$

and therefore

$$V^*(1) \simeq V(i+1-2m).$$

The motive $h^i(X)(m)$ is pure of weight of $w = i - 2m$ and eventually replacing V by $V^*(1)$ one can assume that $w \leq -1$. By the comparison isomorphism of the p -adic Hodge theory [53] $\mathbf{D}_{\text{st}}(H_p^i(X))$ is isomorphic to the log-crystalline cohomology $H_{\log\text{-cris}}^i(X/\mathbf{Q}_p)$. The semisimplicity of the Frobenius action on $H_{\log\text{-cris}}^i(X/\mathbf{Q}_p)$ is a well known open question. The weight monodromy conjecture of Deligne–Jannsen [33] predicts that the absolute values of the eigenvalues of φ acting on $\mathbf{D}_{\text{cris}}(H_p^i(X)) = \mathbf{D}_{\text{st}}(H_p^i(X))^{N=0}$ are $\leq i/2$ and therefore $\mathbf{D}_{\text{cris}}(V)^{\varphi=1}$ should be 0 if $w \leq -1$. If X has good reduction at p we do not need the weight monodromy conjecture and the nullity of $\mathbf{D}_{\text{cris}}(V)^{\varphi=1}$ follows unconditionally from the result of Katz–Messing [36].

Assume that $w = -1$. Then i is odd, $m = \frac{i+1}{2}$ and the isomorphism (30) shows that V is self dual and therefore satisfies **C4b**.

Assume that $w \leq -2$. One expects that

$$(31) \quad H_f^1(\mathbf{Q}, V) = 0 \quad \text{if } w \leq -2.$$

This follows from conjectural properties of the category \mathcal{MM} of mixed motives over \mathbf{Q} . Consider Yoneda groups

$$H^n(\mathbf{Q}, h^i(X)(m)) = \text{Ext}_{\mathcal{MM}}^n(\mathbf{Q}(0), h^i(X)(m)), \quad n = 0, 1$$

and denote by $H_f^1(\mathbf{Q}, h^i(X)(m))$ the subgroup of extensions having "good reduction". It is conjectured that the p -adic realisation functor induces an isomorphism

$$(32) \quad R_p : H_f^1(\mathbf{Q}, h^i(X)(m)) \xrightarrow{\sim} H_f^1(\mathbf{Q}, H_p^i(X)(m)), \quad \forall m \in \mathbf{Z}$$

(see [23]). In particular we should have an isomorphism

$$H_f^1(\mathbf{Q}, h^i(X)(i+1-m)) \xrightarrow{\sim} H_f^1(\mathbf{Q}, V^*(1)).$$

On the other hand from the semisimplicity of the category of pure motives [34] it follows that $H^1(\mathbf{Q}, M) = 0$ for any pure motive M of weight ≥ 0 . In particular, $H^1(\mathbf{Q}, h^i(X)(i+1-m))$ should vanish if $-i-2+2m \geq 0$. Together with (32) this implies **C4b**. To sum up, from the motivic point of view the conditions **C4a** and **C4b** correspond to weight ≤ -2 and weight -1 cases respectively. We consider these two cases separately below.

The weight ≤ -2 case. In this subsection we assume that V satisfies **C1-3** and **C4a**. From **C3** it follows that the exponential map $t_V(\mathbf{Q}_p) \rightarrow H_f^1(\mathbf{Q}_p, V)$ is an isomorphism and we denote by \log_V its inverse. Compositing \log_V with the localization map $H_f^1(\mathbf{Q}, V) \rightarrow H_f^1(\mathbf{Q}_p, V)$ we obtain a map

$$r_V : H_f^1(\mathbf{Q}, V) \rightarrow t_V(\mathbf{Q}_p).$$

One expects that if $V = H_p^i(X)(m)$ with $m \geq i/2 + 1$ this map is related to the syntomic regulator R_{syn} via the commutative diagram

$$\begin{array}{ccc} H_f^1(\mathbf{Q}, h^i(X)(m)) & & \\ \downarrow R_p & \searrow R_{\text{syn}} & \\ H_f^1(\mathbf{Q}, V) & \xrightarrow{r_V} & t_V(\mathbf{Q}_p). \end{array}$$

Our next assumption reflects the hope that the syntomic regulator is an injective map.

C5a) The localization map

$$\text{loc}_p : H_f^1(\mathbf{Q}, V) \rightarrow H_f^1(\mathbf{Q}_p, V)$$

is injective.

In support of this assumption we remark that if $V = H_p^i(X)(m)$ then **C5a)** holds for all $m \neq i/2, i/2 + 1$ if in addition we assume that

$$H^0(\mathbf{Q}_v, V) = 0, \quad \forall v \neq p$$

(and therefore $H_f^1(\mathbf{Q}_v, V) = 0$ for all $v \neq p$) ([33], Lemma 4 and Theorem 3).

Definition (PERRIN-RIOU). Assume that V is a p -adic representation which satisfies the conditions **C1-3)**, **C4a)** and **C5a)**.

i) A (φ, N) -submodule D of $\mathbf{D}_{\text{st}}(V)$ is regular if $D \cap \text{Fil}^0 \mathbf{D}_{\text{st}}(V) = 0$ and the map

$$r_{V,D} : H_f^1(\mathbf{Q}, V) \rightarrow \mathbf{D}_{\text{st}}(V) / (\text{Fil}^0 \mathbf{D}_{\text{st}}(V) + D)$$

induced by r_V is an isomorphism.

ii) Dually, a (φ, N) -submodule D of $\mathbf{D}_{\text{st}}(V^*(1))$ is regular if

$$D + \text{Fil}^0 \mathbf{D}_{\text{st}}(V^*(1)) = \mathbf{D}_{\text{st}}(V^*(1))$$

and the map

$$D \cap \text{Fil}^0 \mathbf{D}_{\text{st}}(V^*(1)) \rightarrow H_f^1(\mathbf{Q}, V)^*$$

induced by the dual map $r_V^* : \text{Fil}^0 \mathbf{D}_{\text{st}}(V^*(1)) \rightarrow H_f^1(V)^*$ is an isomorphism.

Remark. Assume that $H_f^1(\mathbf{Q}, V) = H_f^1(\mathbf{Q}, V^*(1)) = 0$. Then D is regular if the canonical projection $D \rightarrow t_V(\mathbf{Q}_p)$ is an isomorphism of vector spaces and our definition agrees with the definition given in [5].

The weight -1 case. In this subsection we assume that V satisfies the conditions **C1-3)** and **C4b)**. The formula (29) gives

$$\dim_E t_V(\mathbf{Q}_p) = d_+(V).$$

Definition (PERRIN-RIOU). Assume that V is a p -adic representation which satisfies the conditions **C1-3**) and **C4b**). A (φ, N) -submodule D of $\mathbf{D}_{\text{st}}(V)$ is regular if the canonical map $D \rightarrow t_V(\mathbf{Q}_p)$ is an isomorphism.

In the both cases **C4a**) and **C4b**) from (29) it follows immediately that for a regular submodule D one has

$$(33) \quad \dim_E D = d_+(V).$$

Remark. Our definition of a regular module in the weight -1 case is slightly different from Perrin-Riou's definition. If we assume the non-degeneracy of p -adic height pairings $\langle \cdot, \cdot \rangle_{V,D}$ (see Section 4) then our regular modules are regular also in the sense of [47] but the converse is not true. Here we lose in generality to avoid additional technical assumptions.

2.2. Iwasawa cohomology. We keep previous notation and conventions. Let $\mathbf{Q}^{\text{cyc}} = \bigcup_{n \geq 1} \mathbf{Q}(\mu_{p^n})$ and $\Gamma = \text{Gal}(\mathbf{Q}^{\text{cyc}}/\mathbf{Q})$. Then $\Gamma = \Delta \times \Gamma_0$ where $\Delta = \text{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ and $\Gamma_0 = \text{Gal}(\mathbf{Q}^{\text{cyc}}/\mathbf{Q}(\mu_p))$. Set $F_n = \mathbf{Q}(\mu_{p^{n+1}})^\Delta$ and $F_\infty = \bigcup_{n \geq 1} F_n$. The Galois group $\text{Gal}(F_\infty/F)$ is canonically isomorphic to Γ_0 . Let $\Lambda = \mathcal{O}_E[[\Gamma_0]]$ denote the Iwasawa algebra of Γ_0 .

Let V be a p -adic representation of $G_{\mathbf{Q},S}$. Fix a \mathcal{O}_E -lattice T of V stable under the action of $G_{\mathbf{Q},S}$ and consider the Iwasawa cohomology

$$H_{\text{Iw},S}^i(\mathbf{Q}, T) = \varprojlim_{\text{cores}} H_S^i(F_n, T), \quad H_{\text{Iw}}^i(\mathbf{Q}_v, T) = H_{\text{Iw}}^i(\mathbf{Q}_v, T) \otimes_{\mathcal{O}_E} E.$$

The main properties of these groups are summarized below (see also [45]).

i) $H_{\text{Iw},S}^i(\mathbf{Q}, V) = 0$ and $H_{\text{Iw}}^i(\mathbf{Q}_v, T) = 0$ if $i \neq 1, 2$;

ii) If $v \neq p$, then $H_{\text{Iw}}^i(\mathbf{Q}_v, T)$ are finitely-generated Λ -torsion modules. In particular,

$$(34) \quad H_{\text{Iw}}^1(\mathbf{Q}_v, T) \simeq H^1(\mathbf{Q}_v^{\text{ur}}/\mathbf{Q}_v, (\Lambda \otimes_{\mathcal{O}_E} T^{I_v})^I).$$

iii) If $v = p$ then $H_{\text{Iw}}^2(\mathbf{Q}_p, T)$ is a finitely generated Λ -torsion module. Moreover

$$\text{rank}_\Lambda (H_{\text{Iw}}^1(\mathbf{Q}_p, T)) = d, \quad H_{\text{Iw}}^1(\mathbf{Q}_p, T)_{\text{tor}} \simeq H^0(\mathbf{Q}_p(\zeta_{p^\infty}), T).$$

We remark that by local duality $H_{\text{Iw}}^2(\mathbf{Q}_p, T) \simeq H^0(F_{\infty,p}, V^*(1)/T^*(1))$.

The Λ -module structure of $H_{\text{Iw},S}^i(\mathbf{Q}, T)$ depends heavily on the following conjecture formulated by Greenberg [27].

Weak Leopoldt Conjecture. *Let V be a p -adic representation of $G_{\mathbf{Q},S}$ which is potentially semistable at p . Then*

$$H_S^2(F_\infty, V/T) = 0.$$

We have the following result, proved in [47], Proposition 1.3.2.

Proposition 5. *Assume that the weak Leopoldt conjecture holds for V . Then $H_{\text{Iw},S}^2(\mathbf{Q}, T)$ is Λ -torsion and*

$$\text{rank}_\Lambda H_{\text{Iw},S}^1(\mathbf{Q}, T) = d_-(V).$$

Passing to projective limits in the Poitou-Tate exact sequence for V one obtains an exact sequence

$$\begin{aligned} (35) \quad 0 \rightarrow H^2(F_\infty, V^*(1)/T^*(1))^\wedge \rightarrow H_{\text{Iw},S}^1(\mathbf{Q}, T) \rightarrow \bigoplus_{v \in S} H_{\text{Iw}}^1(\mathbf{Q}_v, T) \rightarrow \\ \rightarrow H_S^1(F_\infty, V^*(1)/T^*(1))^\wedge \rightarrow H_{\text{Iw},S}^2(\mathbf{Q}, T) \rightarrow \bigoplus_{v \in S} H_{\text{Iw}}^2(\mathbf{Q}_v, T) \rightarrow \\ \rightarrow H_S^0(F_\infty, V^*(1)/T^*(1))^\wedge \rightarrow 0, \end{aligned}$$

where $(-)^\wedge$ denotes the Pontriagin dual. Therefore if the weak Leopoldt conjecture holds for $V^*(1)$ we have an exact sequence

$$\begin{aligned} 0 \rightarrow H_{\text{Iw},S}^1(\mathbf{Q}, T) \rightarrow \bigoplus_{v \in S} H_{\text{Iw}}^1(\mathbf{Q}_v, T) \rightarrow H_S^1(F_\infty, V^*(1)/T^*(1))^\wedge \rightarrow \\ \rightarrow H_{\text{Iw},S}^2(\mathbf{Q}, T) \rightarrow \bigoplus_{v \in S} H_{\text{Iw}}^2(\mathbf{Q}_v, T) \rightarrow H_S^0(F_\infty, V^*(1)/T^*(1))^\wedge \rightarrow 0. \end{aligned}$$

2.3. The complexes $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ and $\mathbf{R}\Gamma(V, D)$. In this section we construct Selmer complexes $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ and $\mathbf{R}\Gamma(V, D)$ which play the central role in our approach to the Iwasawa theory. Nekovář's book [43] provides a detailed study of Selmer complexes associated to Greenberg's local conditions. For the purposes of this paper one should work with local conditions associated to general (φ, Γ) -submodules of $\mathbf{D}_{\text{rig}}^\dagger(V)$. In this context the general formalism of Selmer complexes was developed by Pottharst in [48] and we refer to *op. cit.* and [49] for further information and details.

Let V be a p -adic representation of the Galois group $G_{\mathbf{Q},S}$ with coefficients in E . We fix a $G_{\mathbf{Q},S}$ -stable lattice T of V . Recall that we denote by $C^\bullet(G_{\mathbf{Q},S}, (T \otimes \Lambda)^l)$ and $C^\bullet(G_{\mathbf{Q}_v}, (T \otimes \Lambda)^l)$ the complexes of continuous cochains of $G_{\mathbf{Q},S}$ and $G_{\mathbf{Q}_v}$ respectively with coefficients in $(T \otimes \Lambda)^l$ and by $\mathbf{R}\Gamma_{\text{Iw},S}(\mathbf{Q}, T)$ and $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_v, T)$ these complexes viewed as objects of $D^b(\Lambda)$. Recall that Shapiro's lemma gives canonical isomorphisms

$$\mathbf{R}^i\Gamma_{\text{Iw},S}(\mathbf{Q}, T) \simeq H_{\text{Iw},S}^i(\mathbf{Q}, T), \quad \mathbf{R}^i\Gamma_{\text{Iw}}(\mathbf{Q}_p, T) \simeq H_{\text{Iw}}^i(\mathbf{Q}_p, T).$$

The derived version of the Poitou–Tate exact sequence for Iwasawa cohomology reads

$$\mathbf{R}\Gamma_{\mathrm{Iw},S}(\mathbf{Q}, T) \rightarrow \bigoplus_{v \in S} \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, T) \rightarrow \mathbf{R}\Gamma(F_\infty, V^*(1)/T^*(1))^\wedge[-2]$$

(see [43], Proposition 8.4.22). Passing to cohomology in this exact triangle one obtains (35).

Let $\bar{V} = V \otimes_{\mathbf{Q}_p} \mathcal{H}^l$. Then

$$\begin{aligned} \mathbf{R}\Gamma_{\mathrm{Iw},S}(\mathbf{Q}, \bar{V}) &\simeq \mathbf{R}\Gamma_{\mathrm{Iw},S}(\mathbf{Q}, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H}, \\ \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, \bar{V}) &\simeq \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H}. \end{aligned}$$

Assume that V is potentially semistable. To each (φ, N) -submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ we associate a complex $\mathbf{R}\Gamma_{\mathrm{Iw}}(V, D)$ which can be seen as a direct generalisation of Selmer complexes determined by Greenberg’s local conditions and studies in Chapters 7-8 of [43]. We define local conditions $U_{\mathrm{Iw},v}^\bullet(V, D)$ ($v \in S$). For $v \neq p$ we set

$$U_{\mathrm{Iw},v}^\bullet(V, D) = \mathbf{R}\Gamma_{\mathrm{Iw},f}(\mathbf{Q}_v, T) \otimes_{\Lambda} \mathcal{H}$$

where

$$\mathbf{R}\Gamma_{\mathrm{Iw},f}(\mathbf{Q}_v, T) = \left[T^{I_v} \otimes \Lambda^l \xrightarrow{\mathrm{Fr}_v - 1} T^{I_v} \otimes \Lambda^l \right].$$

Here I_v denotes the inertia group at v , Fr_v is the geometric Frobenius and the first term is placed in degree 0. Let $H_{\mathrm{Iw},f}^i(\mathbf{Q}_v, T)$ denote the cohomology of this complex. From (34) it follows that $H_{\mathrm{Iw},f}^i(\mathbf{Q}_v, T) = 0$ for $i \neq 1$ and $H_{\mathrm{Iw},f}^1(\mathbf{Q}_v, T) = H_{\mathrm{Iw}}^1(\mathbf{Q}_v, T)$ is Λ -torsion.

Now we define the local condition at p . Let \mathbf{D} be the (φ, Γ) -submodule of $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ associated to D by Theorem 1. Set

$$U_{\mathrm{Iw},p}^\bullet(V, D) = \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{D}) = \left[\mathbf{D}^\Delta \xrightarrow{\psi - 1} \mathbf{D}^\Delta \right].$$

From Theorem 4 it follows that we have a canonical map $U_{\mathrm{Iw},p}^\bullet(V, D) \rightarrow \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_p, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H}$. This gives a diagram in $\mathbf{D}_{\mathrm{coad}}^b(\mathcal{H})$

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathrm{Iw},S}(\mathbf{Q}, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} & \longrightarrow & \bigoplus_{v \in S} \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \\ & & \uparrow \\ & & \bigoplus_{v \in S} U_{\mathrm{Iw},v}^\bullet(V, D) \end{array}$$

and we denote by $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ the associated Selmer complex

$$(36) \quad \mathbf{R}\Gamma_{\text{Iw}}(V, D) = \text{cone} \left(\left(\mathbf{R}\Gamma_{\text{Iw}, S}(\mathbf{Q}, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \right) \bigoplus_{v \in S} \bigoplus U_{\text{Iw}, v}^{\bullet}(V, D) \rightarrow \bigoplus_{v \in S} \mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_v, T) \otimes_{\Lambda}^{\mathbf{L}} \mathcal{H} \right) [1].$$

Now we define the complex $\mathbf{R}\Gamma(V, D)$. For $v \neq p$ let

$$\mathbf{R}\Gamma_f(\mathbf{Q}_v, V) = \left[V \xrightarrow{\text{Fr}_v - 1} V \right]$$

where the first term is placed in degree 0. It is clear that $\mathbf{R}^0\Gamma_f(\mathbf{Q}_v, V) = H^0(\mathbf{Q}_v, V)$ and $\mathbf{R}^1\Gamma_f(\mathbf{Q}_v, V)$ coincides with the group $H_f^1(\mathbf{Q}_v, V)$ defined by (2). Define

$$(37) \quad U_v^{\bullet}(V, D) = \begin{cases} \mathbf{R}\Gamma_f(\mathbf{Q}_v, V), & \text{if } v \neq p \\ \mathbf{R}\Gamma(\mathbf{Q}_p, D), & \text{if } v = p. \end{cases}$$

From Theorem 4 it follows that

$$U_v^{\bullet}(V, D) = U_{\text{Iw}, v}^{\bullet}(V, D) \otimes_{\mathcal{H}}^{\mathbf{L}} E, \quad v \in S.$$

Consider the Selmer complex $\mathbf{R}\Gamma(V, D)$ associated to this data

$$(38) \quad \mathbf{R}\Gamma(V, D) = \text{cone} \left(\mathbf{R}\Gamma_S(\mathbf{Q}, T) \bigoplus_{v \in S} \bigoplus U_v^{\bullet}(V, D) \rightarrow \bigoplus_{v \in S} \mathbf{R}\Gamma(\mathbf{Q}_v, V) \right) [1].$$

We denote by $H_{\text{Iw}}^*(V, D)$ and $H^*(V, D)$ the cohomology of $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ and $\mathbf{R}\Gamma(V, D)$ respectively. The main properties of our Selmer complexes are summarized below.

Proposition 6 (POTTHARST). *i) The complex $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ has cohomology concentrated in degrees $[1, 3]$ consisting of coadmissible \mathcal{H} -modules.*

ii) One has

$$\text{rank}_{\mathcal{H}} H_{\text{Iw}}^1(V, D) = \text{rank}_{\mathcal{H}} H_{\text{Iw}}^2(V, D)$$

Moreover $H_{\text{Iw}}^3(V, D) = (T^(1)^{H_{\mathbf{Q}, S}})^* \otimes_{\Lambda} \mathcal{H}$ where $H_{F, S} = \text{Gal}(\mathbf{Q}^{(S)}/F_{\infty})$.*

iii) The complex $\mathbf{R}\Gamma(V, D)$ has cohomology concentrated in degrees $[0, 3]$ consisting of finite dimensional E -vector spaces and

$$\mathbf{R}\Gamma_{\text{Iw}}(V, D) \otimes_{\mathcal{H}}^{\mathbf{L}} E \simeq \mathbf{R}\Gamma(V, D).$$

In particular, we have canonical exact sequences

$$(39) \quad 0 \rightarrow H_{\text{Iw}}^i(V, D)_{\Gamma} \rightarrow H^i(V, D) \rightarrow H_{\text{Iw}}^{i+1}(V, D)^{\Gamma} \rightarrow 0, \quad i \in \mathbf{N}.$$

iv) *There are canonical dualities*

$$\begin{aligned} \mathrm{Hom}_E(\mathbf{R}\Gamma(V, D), E) &\simeq \mathbf{R}\Gamma(V^*(1), D^\perp)[3], \\ \mathcal{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(V, D)^\iota &\simeq \mathbf{R}\Gamma_{\mathrm{Iw}}(V^*(1), D^\perp)[3]. \end{aligned}$$

In particular, we have canonical exact sequences

$$(40) \quad 0 \rightarrow \mathcal{D}^1 H_{\mathrm{Iw}}^{4-i}(V, D) \rightarrow H_{\mathrm{Iw}}^i(V^*(1), D^\perp)^\iota \rightarrow \mathcal{D}^0 H_{\mathrm{Iw}}^{3-i}(V, D) \rightarrow 0.$$

v) *Assume that D satisfies the following conditions:*

$$\begin{aligned} \mathrm{Fil}^0 D = 0 \quad \text{and} \quad \mathrm{Fil}^0(\mathbf{D}_{\mathrm{st}}(V)/D) &= \mathbf{D}_{\mathrm{st}}(V)/D \\ (\mathbf{D}_{\mathrm{st}}(V)/D)^{\varphi=1, N=0} = 0 \quad \text{and} \quad D/(ND + (p\varphi - 1)D) &= 0. \end{aligned}$$

Then $H^1(\mathbf{Q}_p, \mathbf{D}) = H_f^1(\mathbf{Q}_p, V)$ and $H^1(V, D) = H_f^1(\mathbf{Q}, V)$.

Proof. See [49], Theorem 4.1 and [50], Proposition 3.7. \square

2.4. The Main Conjecture. In this subsection we use the formalism of Selmer complexes to formulate a version of the Main Conjecture of Iwasawa theory. Let M be a pure motive over \mathbf{Q} of weight $w \leq -1$. Since the category of pure motives is semisimple, we can assume that M is simple and $\neq \mathbf{Q}(m)$, $m \in \mathbf{Z}$. Let V denote the p -adic realisation of M . Assume that V is semistable. One expects that for each regular (φ, N) -submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ there exists a p -adic L -function of the form

$$L_p(M, D, s) = f_D(\chi(\gamma_0)^s - 1), \quad f_D \in \mathcal{H}$$

interpolating algebraic parts of special values of the complex L -function $L(M, s)$ (see Section 0.2). Assume that V satisfies the conditions **C1-4** of Section 2.1. We propose the following conjecture.

Main Conjecture. *Let M/\mathbf{Q} be a pure simple motive of weight ≤ -1 . Assume that $M \neq \mathbf{Q}(m)$, $m \in \mathbf{Z}$. Let D be a regular submodule of $\mathbf{D}_{\mathrm{st}}(V)$. Then*

- i) $H_{\mathrm{Iw}}^1(V, D) = 0$,
- ii) $H_{\mathrm{Iw}}^2(V, D)$ is \mathcal{H} -torsion and

$$\mathrm{char}_{\mathcal{H}}(H_{\mathrm{Iw}}^2(V, D)) = (f_D).$$

Remarks. 1) By Proposition 6 the nullity of $H_{\mathrm{Iw}}^1(V, D)$ implies that $H_{\mathrm{Iw}}^2(V, D)$ is \mathcal{H} -torsion.

2) By Proposition 3 any coadmissible \mathcal{H} -torsion module M decomposes into direct product $M \simeq \prod_{i \in I} \mathcal{H}/\mathfrak{p}_i^{n_i}$ and from Lazard's theory [40] it follows that there exists a unique up to multiplication by a unit of $\Lambda[1/p]$ element $f \in \mathcal{H}$ such that $\mathrm{div}(f) = \sum_{i \in I} n_i \mathfrak{p}_i$. The characteristic ideal $\mathrm{char}_{\mathcal{H}} M$ is defined to be the principal ideal generated by f .

3) The condition $M \neq \mathbf{Q}(m)$ implies that $(V^*(1))^{H_{\mathbf{Q},s}} = 0$ and therefore $H_{\text{Iw}}^3(V, D)$ should vanish by Proposition 6, ii). To sum up, we expect that under our assumptions the cohomology of $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ is concentrated in degree 2.

4) Let D^\perp denote the dual regular submodule. One can easily formulate the Main Conjecture for the dual pair $(M^*(1), D^\perp)$. Assume that $H_{\text{Iw}}^1(V, D) = 0$. From (40) it follows that

$$H_{\text{Iw}}^2(V^*(1), D^\perp)^t \simeq H_{\text{Iw}}^2(V, D).$$

This isomorphism reflects the conjectural functional equation relating $L_p(M^*(1), D^\perp, s)$ and $L_p(M, D, s)$.

5) Assume that V is ordinary at p i.e. that the restriction of V on $G_{\mathbf{Q}_p}$ is equipped with an encreasing exhaustive filtration $F^i V$ such that for all $i \in \mathbf{Z}$ the inertia group I_p acts on $\text{gr}^i(V) = F^i V / F^{i+1} V$ by χ^i . In [27], Greenberg works with the Selmer group defined as follows. For each place w of F_∞ including the unique place above p fix a decomposition group H_w for w in $H_{F,S} = \text{Gal}(\mathbf{Q}^{(S)}/F_\infty)$ and denote by I_w its inertia subgroup. Set $A = V/T$ and $F^1 A = F^1 V / F^1 T$. Define

$$(41) \quad H_{\text{Gr}}^1(F_{\infty,w}, A) = \begin{cases} H^1(H_w/I_w, A^{I_w}), & \text{if } w \neq p, \\ \ker(H^1(F_{\infty,p}, A) \rightarrow H^1(I_p, A/F^1 A)) & \text{if } w = p. \end{cases}$$

The Greenberg's Selmer group is defined as follows

$$S(F_\infty, V/T) = \ker \left(H_S^1(F_\infty, A) \rightarrow \bigoplus_{w \in S} \frac{H^1(F_{\infty,w}, A)}{H_{\text{Gr}}^1(F_{\infty,w}, A)} \right).$$

It is well known that each semistable representation is ordinary and we set $D = \mathbf{D}_{\text{st}}(F^1 V)$. Then $\mathbf{R}\Gamma_{\text{Iw}}(\mathbf{D}) = \mathbf{R}\Gamma_{\text{Iw}}(\mathbf{Q}_p, F^1 V) \otimes_{\Lambda[1/p]}^{\mathbf{L}} \mathcal{H}$ and directly from definition (36) it follows that the complex $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ is isomorphic to $\widetilde{\mathbf{R}}\Gamma_{f,\text{Iw}}(F_\infty/\mathbf{Q}, T) \otimes_{\Lambda[1/p]}^{\mathbf{L}} \mathcal{H}$ where $\widetilde{\mathbf{R}}\Gamma_{f,\text{Iw}}(F_\infty/\mathbf{Q}, T)$ denotes Nekovář's Selmer complex associated to the local condition given by $F^1 V$. In [43], Chapter 9, it is proved that under some technical conditions the characteristic ideal of the Pontriagin dual $S(F_\infty, V^*(1)/T^*(1))^\wedge$ of $S(F_\infty, V^*(1)/T^*(1))$ coincides with the characteristic ideal of the second cohomology group $\widetilde{H}_f^2(T)$ of $\widetilde{\mathbf{R}}\Gamma_{f,\text{Iw}}(F_\infty/\mathbf{Q}, T)$. This allows to compare our conjecture to the Main Conjecture of [27].

3. LOCAL STRUCTURE OF p -ADIC REPRESENTATIONS

3.1. Filtration associated to a regular submodule. Let D be a (φ, N) -submodule of $\mathbf{D}_{\text{st}}(V)$ such that $D \cap \text{Fil}^0 \mathbf{D}_{\text{st}}(V) = \{0\}$. We associate to D an

increasing filtration $(D_i)_{i=-2}^2$ on $\mathbf{D}_{\text{st}}(V)$ setting

$$D_i = \begin{cases} 0 & \text{if } i = -2, \\ (1 - p^{-1}\varphi^{-1})D + N(D^{\varphi=1}) & \text{if } i = -1, \\ D & \text{if } i = 0, \\ D + \mathbf{D}_{\text{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) & \text{if } i = 1, \\ \mathbf{D}_{\text{st}}(V) & \text{if } i = 2. \end{cases}$$

It can be easily proved (see[5], Lemma 2.1.5) that $(D_i)_{i=-2}^2$ is the unique filtration on $\mathbf{D}_{\text{st}}(V)$ such that

D1) $D_{-2} = 0$, $D_0 = D$ and $D_2 = \mathbf{D}_{\text{st}}(V)$;

D2) $(\mathbf{D}_{\text{st}}(V)/D_1)^{\varphi=1, N=0} = 0$ and $D_{-1} = (1 - p^{-1}\varphi^{-1})D_{-1} + N(D_{-1})$;

D3) $(D_0/D_{-1})^{\varphi=p^{-1}} = D_0/D_{-1}$ and $(D_1/D_0)^{\varphi=1} = D_1/D_0$.

In addition, for the dual regular submodule D^\perp one has

$$D_i^\perp = \text{Hom}_{\mathbf{Q}_p}(\mathbf{D}_{\text{st}}(V)/D_{-i}, \mathbf{D}_{\text{st}}(\mathbf{Q}_p(1))).$$

Let $(D_i)_{i=-2}^2$ be the filtration associated to D . By (14) it induces a filtration of $\mathbf{D}_{\text{rig}}^\dagger(V)$ which we will denote by $(\mathbf{D}_i)_{i=2}^2$. Define

$$\mathbf{W} = \mathbf{D}_1/\mathbf{D}_{-1}, \quad \mathbf{W}_0 = \text{gr}_0 \mathbf{D}_{\text{rig}}^\dagger(V), \quad \mathbf{W}_1 = \text{gr}_0 \mathbf{D}_{\text{rig}}^\dagger(V).$$

Then we have an exact sequence

$$(42) \quad 0 \rightarrow \mathbf{W}_0 \rightarrow \mathbf{W} \rightarrow \mathbf{W}_1 \rightarrow 0$$

where by **D1-3)**

$$(43) \quad \mathcal{D}_{\text{st}}(\mathbf{W}_0) = \mathcal{D}_{\text{st}}(\mathbf{W}_0)^{\varphi=p^{-1}}, \quad \text{Fil}^0 \mathcal{D}_{\text{st}}(\mathbf{W}_0) = 0,$$

$$(44) \quad \mathcal{D}_{\text{st}}(\mathbf{W}_1) = \mathcal{D}_{\text{st}}(\mathbf{W}_1)^{\varphi=1}.$$

Proposition 7. *Let D be a regular submodule of $\mathbf{D}_{\text{st}}(V)$. Then*

i) *The canonical maps induce inclusions*

$$H^1(\mathbf{D}_{-1}) \subset H_f^1(\mathbf{D}) \subset H^1(\mathbf{D}_1) \subset H^1(\mathbf{Q}_p, V)$$

ii) *One has $H_f^1(\mathbf{D}_{-1}) = H^1(\mathbf{D}_{-1})$.*

iii) *The sequences*

$$0 \rightarrow H^1(\mathbf{D}_{-1}) \rightarrow H^1(\mathbf{D}_1) \rightarrow H^1(\mathbf{W}) \rightarrow 0,$$

$$0 \rightarrow H^1(\mathbf{D}_{-1}) \rightarrow H_f^1(\mathbf{D}_1) \rightarrow H_f^1(\mathbf{W}) \rightarrow 0$$

are exact.

Proof. 1) By **D2**) one has

$$\mathrm{Hom}(D_{-1}, \mathbf{D}_{\mathrm{st}}(\mathbf{Q}_p(1)))^{\varphi=1, N=0} = 0$$

and by the Poincaré duality

$$(45) \quad H^2(\mathbf{D}_{-1}) \simeq H^0(\mathbf{D}_{-1}^*(\chi))^* = 0.$$

Now (8) and (23) implies that $H_f^1(\mathbf{D}_{-1}) = H^1(\mathbf{D}_{-1})$ and ii) is proved.

2) The exact sequence

$$0 \rightarrow \mathbf{D}_1 \rightarrow \mathbf{D}_{\mathrm{rig}}^\dagger(V) \rightarrow \mathrm{gr}_2 \mathbf{D}_{\mathrm{rig}}^\dagger(V) \rightarrow 0$$

gives an exact sequence

$$0 \rightarrow H^0(\mathrm{gr}_2 \mathbf{D}_{\mathrm{rig}}^\dagger(V)) \rightarrow H^1(\mathbf{D}_1) \rightarrow H^1(\mathbf{Q}_p, V).$$

From **D2**) it follows that

$$H^0(\mathrm{gr}_2 \mathbf{D}_{\mathrm{rig}}^\dagger(V)) = \mathrm{Fil}^0(\mathbf{D}_{\mathrm{st}}(V)/D_1)^{\varphi=1, N=0} = 0$$

and the injectivity of $H^1(\mathbf{D}_1) \rightarrow H^1(\mathbf{Q}_p, V)$ is proved. Since $\mathrm{Fil}^0 \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) = 0$ and $\mathcal{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=1} = \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$ by **C3**), the exact sequence (22) induces a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) & \xrightarrow{\exp} & H_f^1(\mathbf{D}) \\ \downarrow & & \downarrow \\ t_V(\mathbf{Q}_p) & \xrightarrow{\exp} & H_f^1(\mathbf{Q}_p, V). \end{array}$$

where the exponential maps and the left vertical map are isomorphisms. Thus $H_f^1(\mathbf{D}) \rightarrow H_f^1(\mathbf{Q}_p, V)$ is an injection. The same argument together with ii) proves that $H^1(\mathbf{D}_{-1}) \subset H_f^1(\mathbf{D})$. This implies i).

3) Since the sequence

$$0 \rightarrow H^1(\mathbf{D}_{-1}) \rightarrow H^1(\mathbf{D}_1) \rightarrow H^1(\mathbf{W}) \rightarrow H^2(\mathbf{D}_{-1})$$

is exact, iii) follows from ii) and (45). \square

Assume now that the canonical projection $\mathcal{D}_{\mathrm{st}}(\mathbf{D}) \rightarrow t_V(\mathbf{Q}_p)$ is an isomorphism i.e. that

$$(46) \quad \mathbf{D}_{\mathrm{st}}(V) = D \oplus \mathrm{Fil}^0 \mathbf{D}_{\mathrm{st}}(V)$$

as E -vector spaces. In this case the structure of \mathbf{W} can be completely determined if we make the following additional assumption (see [28], [5]).

U) The (φ, Γ) -module $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ has no saturated *crystalline* subquotient U sitting in a non split exact sequence of the form

$$(47) \quad 0 \rightarrow \mathcal{R}_E(|x|x^k) \rightarrow U \rightarrow \mathcal{R}_E(x^m) \rightarrow 0.$$

We remark that if $k \leq m$, then $H_f^1(\mathcal{R}_E(|x|x^{k-m})) = 0$ and there is no non trivial crystalline extension (47). If $k > m$, it follows from (23) that $\dim_E H_f^1(\mathcal{R}_E(|x|x^{k-m})) = 1$ and therefore there exists a unique (up to isomorphism) crystalline (φ, Γ) -module U of the form (47).

Proposition 8. *Let D be a (φ, N) -submodule of $\mathbf{D}_{\text{st}}(V)$ which satisfies (46). Assume that the condition **U**) holds. Then*

i) *There exists a unique decomposition*

$$(48) \quad \mathbf{W} \simeq \mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \mathbf{M}$$

where \mathbf{A}_0 and \mathbf{A}_1 are direct summands of \mathbf{W}_0 and \mathbf{W}_1 of ranks $\dim_E H^0(\mathbf{W}^*(\chi))$ and $\dim_E H^0(\mathbf{W})$ respectively. Moreover, \mathbf{M} is inserted in an exact sequence

$$0 \rightarrow \mathbf{M}_0 \xrightarrow{f} \mathbf{M} \xrightarrow{g} \mathbf{M}_1 \rightarrow 0$$

where $\mathbf{W}_0 \simeq \mathbf{A}_0 \oplus \mathbf{M}_0$, $\mathbf{W}_1 \simeq \mathbf{A}_1 \oplus \mathbf{M}_1$ and $\text{rank}(\mathbf{M}_0) = \text{rank}(\mathbf{M}_1)$.

ii) *One has*

$$\dim_E H^1(\mathbf{M}) = 2e, \quad \dim_E H_f^1(\mathbf{M}) = e, \quad \text{where } e = \text{rank}(\mathbf{M}_0) = \text{rank}(\mathbf{M}_1).$$

iii) *Consider the exact sequence*

$$0 \rightarrow H^0(\mathbf{M}_1) \xrightarrow{\delta_0} H^1(\mathbf{M}_0) \xrightarrow{f_1} H^1(\mathbf{M}) \xrightarrow{g_1} H^1(\mathbf{M}_1) \xrightarrow{\delta_1} H^2(\mathbf{M}_0) \rightarrow 0.$$

Then $H^1(\mathbf{M}_0) \simeq \text{Im}(\delta_0) \oplus H_f^1(\mathbf{M}_0)$, $\text{Im}(f_1) = H_f^1(\mathbf{M})$ and $H^1(\mathbf{M}_1) \simeq \text{Im}(g_1) \oplus H_f^1(\mathbf{M}_1)$.

Proof. See [5], Proposition 2.1.7 and Lemma 2.1.8. \square

3.2. The weight ≤ -2 case. We return to the study of the cohomology of p -adic representations. Let V is the p -adic realisation of a pure simple motive $M \neq \mathbf{Q}(m)$. In this subsection we assume that V satisfies the conditions **C1-4a**) of Section 2.1. Let D be a regular submodule of $\mathbf{D}_{\text{st}}(V)$ and \mathbf{D} the associated (φ, Γ) -submodule of $\mathbf{D}_{\text{rig}}^\dagger(V)$. In the remainder of this paper we write $H^1(\mathbf{D})$ instead $H^1(\mathbf{Q}_p, \mathbf{D})$ to simplify notation.

Define

$$H_{f, \{p\}}^1(\mathbf{Q}, V) = \ker \left(H_S^1(\mathbf{Q}, V) \rightarrow \bigoplus_{v \in S - \{p\}} \frac{H^1(\mathbf{Q}_v, V)}{H_f^1(\mathbf{Q}_v, V)} \right).$$

Then (28) gives an exact sequence

$$(49) \quad 0 \rightarrow H_f^1(\mathbf{Q}, V) \rightarrow H_{f, \{p\}}^1(\mathbf{Q}, V) \rightarrow \frac{H^1(\mathbf{Q}_p, V)}{H_f^1(\mathbf{Q}_p, V)} \rightarrow 0.$$

From the regularity of D we have a decomposition

$$H_f^1(\mathbf{Q}_p, V) = H_f^1(\mathbf{Q}, V) \oplus H_f^1(\mathbf{D})$$

and therefore the restriction map induces an isomorphism

$$H_{f, \{p\}}^1(\mathbf{Q}, V) \simeq \frac{H^1(\mathbf{Q}_p, V)}{H_f^1(\mathbf{D})}.$$

Let $H_D^1(\mathbf{Q}, V)$ denote the inverse image of $H^1(\mathbf{D}_1)/H_f^1(\mathbf{D})$ under this isomorphism. By Proposition 5 iii) we have an injection

$$(50) \quad \kappa_D : H_D^1(\mathbf{Q}, V) \hookrightarrow H^1(\mathbf{W})$$

Since by **D3**) $\mathcal{D}_{\text{cris}}(\mathbf{W}_0)^{\varphi=p^{-1}} = \mathcal{D}_{\text{cris}}(\mathbf{W}_0)$ and $\text{Fil}^0 \mathcal{D}_{\text{cris}}(\mathbf{W}_0) = 0$, the results of Section 1.8 give a canonical decomposition

$$(51) \quad H^1(\mathbf{W}_0) = H_f^1(\mathbf{W}_0) \oplus H_c^1(\mathbf{W}_0).$$

Proposition 9. *Assume that the weak Leopoldt conjecture holds for $V^*(1)$ and that*

- a) $H^0(\mathbf{W}) = 0$.
- b) *The map $H_c^1(\mathbf{W}_0) \rightarrow H^1(\mathbf{W})$ is injective.*
- c) *One has*

$$(52) \quad \text{Im}(\kappa_D) \cap H_c^1(\mathbf{W}_0) = \{0\} \quad \text{in} \quad H^1(\mathbf{W}).$$

Then $H_{\text{Iw}}^1(V, D) = 0$ and therefore $H_{\text{Iw}}^2(V, D)$ is \mathcal{H} -torsion.

Proof. The reader can compare this proof to the proof of Theorem 5.1.3 of [7]. We will use the following elementary lemma.

Lemma 1. *Let A and B be two submodules of a finitely-generated free \mathcal{H} -module M . Assume that the natural maps $A_{\Gamma_1} \rightarrow M_{\Gamma_1}$ and $B_{\Gamma_1} \rightarrow M_{\Gamma_1}$ are both injective. Then $A_{\Gamma_1} \cap B_{\Gamma_1} = \{0\}$ implies that $A \cap B = \{0\}$.*

Proof. This is Lemma 5.1.4.1 of [7]. □

We prove Proposition 9. Because the weak Leopoldt conjecture holds for $V^*(1)$ the group $H_{\text{Iw}, S}^1(\mathbf{Q}, T)$ injects into $\bigoplus_{v \in S} H_{\text{Iw}}^1(\mathbf{Q}_v, T)$. Since $H_{\text{Iw}, f}^1(\mathbf{Q}_v, T) = H_{\text{Iw}}^1(\mathbf{Q}_v, T)$ for $v \neq p$, by the definition of the complex $\mathbf{R}\Gamma_{\text{Iw}}(V, D)$ one has

$$H_{\text{Iw}}^1(V, D) = (H_{\text{Iw}, S}^1(\mathbf{Q}, T) \otimes_{\Lambda} \mathcal{H}) \cap H_{\text{Iw}}^1(\mathbf{D}) \quad \text{in} \quad H_{\text{Iw}}^1(\mathbf{Q}_p, T) \otimes_{\Lambda} \mathcal{H}.$$

The Λ -torsion part of $H_{\text{Iw}, S}^1(\mathbf{Q}, T)$ is isomorphic to $T^{H_{\mathbf{Q}, S}} = 0$ and therefore $A = H_{\text{Iw}, S}^1(\mathbf{Q}, T) \otimes_{\Lambda} \mathcal{H}$ is a free \mathcal{H} -module. Moreover $A_{\Gamma} \subset H_{f, \{p\}}^1(\mathbf{Q}, V)$. Set $B = H_{\text{Iw}}^1(\mathbf{D})/H_{\text{Iw}}^1(\mathbf{D})_{\mathcal{H}\text{-tor}}$. The \mathcal{H} -torsion part of $H_{\text{Iw}}^1(\mathbf{D})$ is contained

in $T^{H\mathbf{Q}_p} \otimes_{\Lambda} \mathcal{H}$ and $\dim_E \left(V^{H\mathbf{Q}_p} \right)_{\Gamma} = \dim_E \left(V^{H\mathbf{Q}_p} \right)^{\Gamma} = 0$ and therefore $B_{\Gamma} = H_{\text{Iw}}^1(\mathbf{D})_{\Gamma}$.

We will prove that $B_{\Gamma} = H_{\text{Iw}}^1(\mathbf{D})_{\Gamma}$ injects into $H^1(\mathbf{D}_1) \subset H^1(\mathbf{Q}_p, V)$. The exact sequence

$$0 \rightarrow \mathbf{D} \rightarrow \mathbf{D}_1 \rightarrow \mathbf{W}_1 \rightarrow 0$$

gives rise to an exact sequence $0 \rightarrow H^0(\mathbf{W}_1) \rightarrow H^1(\mathbf{D}) \rightarrow H^1(\mathbf{D}_1)$ and therefore it is sufficient to show that

$$H^0(\mathbf{W}_1) \cap H_{\text{Iw}}^1(\mathbf{D})_{\Gamma} = 0 \quad \text{in} \quad H^1(\mathbf{D}).$$

Set $Z = H^0(\mathbf{W}_1) \cap H_{\text{Iw}}^1(\mathbf{D})_{\Gamma}$. Let $f : H^1(\mathbf{D}) \rightarrow H^1(\mathbf{W}_0)$ denote the map induced by the natural projection $\mathbf{D} \rightarrow \mathbf{W}_0$. Since $H^0(\mathbf{W}) = 0$, the exact sequence $0 \rightarrow \mathbf{W}_0 \rightarrow \mathbf{W} \rightarrow \mathbf{W}_1 \rightarrow 0$ induces an injection $H^0(\mathbf{W}_1) \hookrightarrow H^1(\mathbf{W}_0)$. Thus $Z \cap \ker(f) = \{0\}$. On the other hand, by Proposition 4

$$f(H_{\text{Iw}}^1(\mathbf{D})_{\Gamma}) \subset H_{\text{Iw}}^1(\mathbf{W}_0) = H_c^1(\mathbf{W}_0).$$

From the exact sequence

$$0 \rightarrow H^0(\mathbf{W}_1) \hookrightarrow H^1(\mathbf{W}_0) \rightarrow H^1(\mathbf{W})$$

and the injectivity of $H_c^1(\mathbf{W}_0) \rightarrow H^1(\mathbf{W})$ we obtain that $H_c^1(\mathbf{W}_0) \cap H^0(\mathbf{W}_1) = \{0\}$ and therefore $f(Z) = 0$. This proves that $Z = \{0\}$ and the injectivity of the map $H_{\text{Iw}}^1(\mathbf{D})_{\Gamma} \rightarrow H^1(\mathbf{D}_1)$.

To complete the proof of Proposition, by Lemma 1 it is enough to show that

$$(53) \quad H_{f, \{p\}}^1(\mathbf{Q}, V) \cap H_{\text{Iw}}^1(\mathbf{D})_{\Gamma} = 0 \quad \text{in} \quad H^1(\mathbf{Q}_p, V).$$

Since $H_{\text{Iw}}^1(\mathbf{D})_{\Gamma} \subset H^1(\mathbf{D}_1)$ and the map κ_D is injective, (53) is equivalent to

$$H_D^1(\mathbf{Q}, V) \cap H_c^1(\mathbf{W}_0) = 0$$

and the Proposition is proved. \square

We want to discuss the relationship of this result with the phenomenon of trivial zeros of p -adic L -functions studied in [28], [5], [7]. We consider two cases.

a) The case $\mathbf{D}_{\text{st}}(V)^{\varphi=1} = 0$. Assume in addition that $\mathbf{D}_{\text{st}}(V)^{\varphi=1} = 0$. Then $\mathbf{D}_1 = \mathbf{D}$ and $\mathbf{W} = \mathbf{W}_0$. By Proposition 7 iii) one has

$$H_D^1(\mathbf{Q}, V) \simeq H^1(\mathbf{D})/H_f^1(\mathbf{D}) \simeq H^1(\mathbf{W}_0)/H_f^1(\mathbf{W}_0).$$

Consider the commutative diagram

$$\begin{array}{ccc}
 \mathcal{D}_{\text{cris}}(\mathbf{W}_0) & \xrightarrow{i_{D,f}} & H_f^1(\mathbf{W}_0) \\
 \rho_{D,f} \uparrow & & \uparrow p_{D,f} \\
 H_D^1(\mathbf{Q}, V) & \longrightarrow & H^1(\mathbf{W}_0) \\
 \rho_{D,c} \downarrow & & \downarrow p_{D,c} \\
 \mathcal{D}_{\text{cris}}(\mathbf{W}_0) & \xrightarrow{i_{D,c}} & H_c^1(\mathbf{W}_0),
 \end{array}$$

where $p_{D,f}$ and $p_{D,c}$ are projections given by (51) and $\rho_{D,f}$ and $\rho_{D,c}$ are defined as the unique maps making this diagram commute.

Definition (see [7]). *Let V be a p -adic representation which satisfies the conditions **C1-4a**). Assume that $\mathbf{D}_{\text{st}}(V)^{\varphi=1} = 0$. Let D be a regular submodule of $\mathbf{D}_{\text{st}}(V)$. The determinant*

$$\ell(V, D) = \det \left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\text{cris}}(\mathbf{W}_0) \right)$$

will be called the \mathcal{L} -invariant associated to V and D .

b) The case $H_f^1(\mathbf{Q}, V) = H_f^1(\mathbf{Q}, V^*(1)) = 0$. Assume that

$$(54) \quad H_f^1(\mathbf{Q}, V) = H_f^1(\mathbf{Q}, V^*(1)) = 0$$

and that in addition V satisfies

M) The condition **U**) of Section 3.1 holds and in the decomposition (48) $\mathbf{A}_0 = \mathbf{A}_1 = 0$.

Note that the typical example we have in mind is $W_f(k)$ where W_f is the p -adic representation associated to a split multiplicative newform f of weight $2k$ on $\Gamma_0(N)$ with $p \mid N$.

From Proposition 8 one has $H_f^1(\mathbf{M}_0) = H_f^1(\mathbf{M})$ and $H^1(\mathbf{M})/H_f^1(\mathbf{M}_0) \simeq H^1(\mathbf{M}_1)/H_f^1(\mathbf{M}_1)$. Thus $H_D^1(\mathbf{Q}, V) \simeq H^1(\mathbf{M}_1)/H_f^1(\mathbf{M}_1)$ is a E -vector space of dimension $e = \text{rank}(\mathbf{M}_0) = \text{rank}(\mathbf{M}_1)$ and one has a commutative diagram

$$\begin{array}{ccccccc}
 (55) & 0 & \longrightarrow & H^0(\mathbf{M}_1) & \xrightarrow{\delta_0} & H^1(\mathbf{M}_0) & \xrightarrow{f_1} & H^1(\mathbf{M}) & \xrightarrow{g_1} & H^1(\mathbf{M}_1) & \xrightarrow{\delta_1} & H^2(\mathbf{M}_0) & \longrightarrow & 0 \\
 & & & & & & & \uparrow \kappa_D & & \nearrow \bar{\kappa}_D & & & & \\
 & & & & & & & H_D^1(\mathbf{Q}, V) & & & & & &
 \end{array}$$

where the map $\bar{\kappa}_D$ is injective. As $H^0(\mathbf{M}) = H^2(\mathbf{M}) = 0$, the upper row is exact. This implies that $\text{Im}(g_1)$ is a E -vector space of dimension e . Thus $\text{Im}(\bar{\kappa}_D) = \text{Im}(g_1)$ and again one has a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{D}_{\text{cris}}(\mathbf{M}_1) & \xrightarrow{i_{D,f}} & H_f^1(\mathbf{M}_1) \\
 & \uparrow \rho_{M,f} & & & \uparrow p_{D,f} \\
 H_D^1(\mathbf{Q}, V) & \xrightarrow{\sim} & \text{Im}(g_1) & \longrightarrow & H^1(\mathbf{M}_1) \\
 & \downarrow \rho_{M,c} & & & \downarrow p_{D,c} \\
 & & \mathcal{D}_{\text{cris}}(\mathbf{M}_1) & \xrightarrow{i_{D,c}} & H_c^1(\mathbf{M}_1)
 \end{array}$$

Definition (see [5]). *Let V be a p -adic representation which satisfies the conditions **C1-3**) and (54). Let D be a regular submodule of $\mathbf{D}_{\text{st}}(V)$. Assume that the condition **M**) holds. The determinant*

$$\ell(V, D) = \det \left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\text{cris}}(\mathbf{M}_1) \right)$$

will be called the ℓ -invariant associated to V and D .

In particular, in this case the ℓ -invariant is local, i.e. depends only on the restriction of the representation V on a decomposition group at p .

Remark. In [5] the ℓ -invariant is defined in a slightly more general situation where only \mathbf{W}_0 vanishes. We do not include it here to avoid additional technical complications in the formulation of our results.

The following conjecture was formulated in [5] and [7].

Conjecture (EXTRA ZERO CONJECTURE). *Let V be a p -adic representation which satisfies the conditions **C1-4a**). Let D be a regular subspace of $\mathbf{D}_{\text{st}}(V)$ and let $e = \text{rank}(\mathbf{W}_0)$. Then in both cases **a**) and **b**) the p -adic L -function $L_p(V, D, s)$ has a zero of order e at $s = 0$ and*

$$\lim_{s \rightarrow 0} \frac{L_p(V, D, 0)}{s^e} = \ell(V, D) \mathcal{E}^+(V, D) \Omega_p(M, D) \frac{L(M, 0)}{\Omega_\infty(M)}$$

where $\mathcal{E}^+(V, D)$ is obtained from $\mathcal{E}(V, D)$ by excluding zero factors.

Recall (see Section 0.2) that $\Omega_p(M, D)$ denote the determinant of the regulator map $r_{V,D}$. Note that $\Omega_p(M, D) = 1$ if $H_f^1(\mathbf{Q}, V) = H_f^1(\mathbf{Q}, V^*(1)) = 0$. We refer to [5], [6] [7] for precise formulation of this conjecture and a survey of known cases. Note that the non vanishing of $\ell(V, D)$ is a difficult open problem which is solved only for Dirichlet motives $\mathbf{Q}(\eta)$ [21] and for

elliptic curves [1]. The following result shows that it is closely related to the expected vanishing of $H_{\text{Iw}}^1(V, D)$.

Proposition 10. *Let V be a p -adic representation which satisfies the conditions **C1-3**). Assume that the weak Leopoldt conjecture holds for $V^*(1)$. Then in the both cases **a**) and **b**) above the non vanishing of $\ell(V, D)$ implies that $H_{\text{Iw}}^1(V, D) = 0$ and therefore $H_{\text{Iw}}^2(V, D)$ in \mathcal{H} -torsion.*

Proof. In the case **a**) $\mathbf{W}_0 = \mathbf{W}$ and the statement follows directly from Proposition 9 and the definition of the ℓ -invariant. In the case **b**) the diagram (55) together with Proposition 8 show that the injectivity of $H_c^1(\mathbf{M}_0) \rightarrow H^1(\mathbf{M})$ and the condition (52) are both equivalent to the following condition

$$\text{Im}(\delta_0) \cap H_c^1(\mathbf{M}_0) = \{0\}.$$

The statement follows now from [5], Proposition 2.2.4 where it is proved that $\ell(V, D)$ can be computed as the slope of the map $\delta_0 : H^0(\mathbf{M}_1) \rightarrow H^1(\mathbf{M})$ with respect to the decomposition $H^1(\mathbf{M}) \simeq H_f^1(\mathbf{M}) \oplus H_c^1(\mathbf{M})$. \square

4. THE p -ADIC HEIGHT PAIRING

4.1. The extended Selmer group. Let V be a p -adic representation of $G_{\mathbf{Q}, S}$ which satisfies the conditions **C1-3**) of Section 2.1. Fix a submodule D of $\mathbf{D}_{\text{st}}(V)$ such that $\mathbf{D}_{\text{st}}(V) = D \oplus \text{Fil}^0 \mathbf{D}_{\text{st}}(V)$. We will always assume that the condition **M**) of Section 3.2 holds for D . Then by Proposition 8 one has an exact sequence

$$0 \rightarrow \mathbf{M}_0 \rightarrow \mathbf{M} \rightarrow \mathbf{M}_1 \rightarrow 0$$

where $\mathcal{D}_{\text{cris}}(\mathbf{M}_0) = D/D_{-1}$, $\mathcal{D}_{\text{cris}}(\mathbf{M}_1) = D_1/D$ and $\text{rank}(\mathbf{M}_0) = \text{rank}(\mathbf{M}_1)$.

Proposition 11. *There exists an exact sequence*

$$(56) \quad 0 \rightarrow H^0(\mathbf{M}_1) \rightarrow H^1(V, D) \rightarrow H_f^1(\mathbf{Q}, V) \rightarrow 0,$$

where $H_f^1(\mathbf{Q}, V)$ is the Bloch–Kato’s Selmer group and $\dim_E H^0(\mathbf{M}_1) = \text{rank}(\mathbf{M}_1)$.

Proof. By the definition of $\mathbf{R}\Gamma(V, D)$ the group $H^1(V, D)$ is the kernel of the map

$$H_S^1(\mathbf{Q}, V) \oplus \left(\bigoplus_{v \in S - \{p\}} H_f^1(\mathbf{Q}_v, V) \right) \oplus H^1(\mathbf{D}) \rightarrow \bigoplus_{v \in S} H^1(\mathbf{Q}_v, V).$$

The Proposition follows directly from this description of $H^1(V, D)$ together with the following facts

$$\text{a) } H^0(\mathbf{M}_1) = \ker(H^1(\mathbf{D}) \rightarrow H^1(\mathbf{Q}_p, V));$$

$$\text{b) } \text{Im}(H^1(\mathbf{D}) \rightarrow H^1(\mathbf{Q}_p, V)) = H_f^1(\mathbf{Q}_p, V).$$

The proof of a) and b) can be extracted from the construction of the ℓ -invariant in [5], Section 2.2.1 but we recall the arguments for reader's convenience. Consider the exact sequence

$$0 \rightarrow \mathbf{D}_1 \rightarrow \mathbf{D}_{\text{rig}}^\dagger(V) \rightarrow \text{gr}_2 \mathbf{D}_{\text{rig}}^\dagger(V) \rightarrow 0.$$

In the proof of Proposition 7 we saw that $H^0(\text{gr}_2 \mathbf{D}_{\text{rig}}^\dagger(V)) = 0$ and therefore the map $H^1(\mathbf{D}_1) \rightarrow H^1(\mathbf{Q}_p, V)$ is injective. Taking the long cohomology sequence associated to

$$0 \rightarrow \mathbf{D} \rightarrow \mathbf{D}_1 \rightarrow \mathbf{M}_1 \rightarrow 0$$

we obtain

$$0 \rightarrow H^0(\mathbf{M}_1) \rightarrow H^1(\mathbf{D}) \rightarrow H^1(\mathbf{D}_1)$$

and a) is proved.

Using (22) it is not difficult to show that $H_f^1(\text{gr}_2 \mathbf{D}_{\text{rig}}^\dagger(V)) = 0$ and therefore $H_f^1(\mathbf{D}_1) \simeq H_f^1(\mathbf{Q}_p, V)$. Consider the exact sequence

$$0 \rightarrow \mathbf{D}_{-1} \rightarrow \mathbf{D}_1 \rightarrow \mathbf{M} \rightarrow 0.$$

Since $H^0(\mathbf{M}) = 0$ and $H_f^1(\mathbf{D}_{-1}) = H^1(\mathbf{D}_{-1})$, by [?], Corollary 1.4.6 one has an exact sequence

$$0 \rightarrow H^1(\mathbf{D}_{-1}) \rightarrow H_f^1(\mathbf{D}_1) \rightarrow H_f^1(\mathbf{M}) \rightarrow 0.$$

On the other hand, from Proposition 8

$$\text{Im}(H^1(\mathbf{M}_0) \rightarrow H^1(\mathbf{M})) = H_f^1(\mathbf{M}).$$

Thus $\text{Im}(H^1(\mathbf{D}) \rightarrow H^1(\mathbf{D}_1)) = H_f^1(\mathbf{D}_1)$ and b) is proved. \square

Definition. We will call $H^1(V, D)$ the extended Selmer group associated to D .

Remarks 1) Extended Selmer groups associated to ordinary local conditions were studied in [42], [43].

2) If $\mathbf{M} = 0$ the group $H^1(V, D)$ coincides with the Bloch–Kato's Selmer group. One expects that if D is regular, the appearance of $H^0(\mathbf{M}_1)$ in the short exact sequence (56) reflects the presence of extra-zeros of the p -adic L -function $L_p(V, D, s)$. We study this question in [8].

4.2. The p -adic height pairing. In [43], Nekovář found a new construction of the p -adic height pairing on extended Selmer groups defined by Greenberg's local conditions. In this section we follow his approach *verbatim* working with local conditions defined by (φ, N) -submodules. We keep notation and conventions of Section 4.1. Set $A = \mathcal{H}/(X^2)$. Then also $A = E[X]/(X^2)$ and one has an exact sequence

$$(57) \quad 0 \rightarrow E \xrightarrow{X} A \rightarrow E \rightarrow 0.$$

Set $V_A = V \otimes_E A^t$ and $D_A = D \otimes_E A^t$ and consider the complex $\mathbf{R}\Gamma(V_A, D_A)$. The sequence (57) induces a distinguished triangle

$$\mathbf{R}\Gamma(V_A, D) \rightarrow \mathbf{R}\Gamma(V_A, D_A) \rightarrow \mathbf{R}\Gamma(V, D)$$

which gives the coboundary map

$$H^1(V, D) \xrightarrow{\beta} H^2(V, D).$$

Let $\mathbf{D}^\perp \subset \mathbf{D}_{\text{st}}(V^*(1))$ denote the dual submodule.

Definition. Let V be a p -adic representation which satisfies the conditions **C1-3**) and let D be a (φ, N) -submodule of $\mathbf{D}_{\text{st}}(V)$ such that the condition **M**) holds for D . We define the p -adic height pairing associated to D as the bilinear map

$$(58) \quad \langle \cdot, \cdot \rangle_{V, D} : H^1(V, D) \times H^1(V^*(1), D^\perp) \rightarrow E$$

given by $\langle x, y \rangle_{V, D} = \beta(x) \cup y$ where $\cup : H^2(V, D) \times H^1(V^*(1), D^\perp) \rightarrow E$ denotes the duality defined in Proposition 6.

Remarks 1) In the ordinary setting it is possible to work over \mathbf{Z}_p and Nekovář's descent machinery gives very general formulas of the Birch and Swinnerton-Dyer type. In the general setting we are forced to work over \mathbf{Q}_p if we want to use the theory of (φ, Γ) -modules.

2) The pairing (58) will be studied in detail in [8]. In particular we compare our construction to the p -adic height pairing constructed by Nekovář in [42] and relate it to universal norms.

The following result can be seen as an analog of Proposition 10 in the weight -1 case.

Proposition 12. Let V be a p -adic representation which satisfies the conditions **C1-3**) and does not contain subrepresentations of the form $\mathbf{Q}_p(m)$. Let D be a (φ, N) -submodule of $\mathbf{D}_{\text{st}}(V)$ such that the condition **M**) holds for D . Assume that

- a) The weak Leopoldt conjecture holds for $V^*(1)$.
- b) The pairing $\langle \cdot, \cdot \rangle_{V, D}$ is non degenerate.

Then $H_{\text{Iw}}^1(V, D) = 0$ and therefore $H_{\text{Iw}}^2(V, D)$ is \mathcal{H} -torsion.

Proof. If the p -adic height pairing is non degenerate, the map β is injective. The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H} & \xrightarrow{X} & \mathcal{H} & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E & \xrightarrow{X} & A & \longrightarrow & E \longrightarrow 0 \end{array}$$

gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{Iw}}^1(V, D)_{\Gamma} & \longrightarrow & H^1(V, D) & \longrightarrow & H_{\text{Iw}}^2(V, D)^{\Gamma} \\ & & \downarrow & & \downarrow = & & \downarrow \\ & & H^1(V_A, D_A) & \longrightarrow & H^1(V, D) & \xrightarrow{\beta} & H^2(V, D). \end{array}$$

Thus $H_{\text{Iw}}^1(V, D)_{\Gamma} = 0$. On the other hand, as the weak Leopoldt conjecture holds for $V^*(1)$ and $V^{H_{\mathbf{Q}, S}} = 0$, the $\Lambda[1/p]$ -module $H_{\text{Iw}, S}^1(\mathbf{Q}, V)$ is free and injects into $H_{\text{Iw}}^1(\mathbf{Q}_p, V)$. Therefore, as in the proof of Proposition 9 one has

$$H_{\text{Iw}}^1(V, D) = (H_{\text{Iw}, S}^1(\mathbf{Q}, T) \otimes_{\Lambda} \mathcal{H}) \cap H_{\text{Iw}}^1(\mathbf{D}).$$

This implies that $H_{\text{Iw}}^1(V, D)$ is \mathcal{H} -free and the vanishing of $H_{\text{Iw}}^1(V, D)_{\Gamma}$ gives $H_{\text{Iw}}^1(V, D) = 0$. \square

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